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A SOLUTION OF THE NAVIER-STOKES EQUATION FOR SOURCE AND SINK FLOWS OF A VISCOUS HEAT-CONDUCTING COMPRESSIBLE FLUID

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SUMMARY

A solution of the Navier-Stokes equation for source and sink flows of a viscous, heat-conducting, compressible fluid is given for the case of constant total flow energy. For the satisfaction of the condition of constant total flow energy, a certain Prandtl number is required. Aside from this more obvious requirement, the selection of a certain ratio of the first and second viscosity coefficient is necessary. The nature of the general solutions for flow with arbitrary Prandtl number and with heat addition is discussed. Furthermore, the manner is discussed in which the familiar heat-conduction effects combined with the peculiar viscous effects solely due to compressibility, sometimes called the longitudinal viscous effects, influence the flow through a curved minimum section joined to a sink flow. A discussion of the second viscosity coefficient from the gas-dynamic approach is also given.

INTRODUCTION

With the advent of flight at extreme altitudes, the study of high-speed viscous flows has become of practical importance. Since for high-speed flows the compressibility effects are large, the viscous effects solely due to compressibility also called the longitudinal viscous effects have to be investigated in addition to the transverse viscous effects which are associated with boundary layers.

Since the transverse viscous effects and the longitudinal viscous effects in one dimension are well-known, there remain to be investigated the longitudinal viscous effects in more than one dimension and their interaction with the transverse viscous effects. Recently, an extensive study has been made by Lagerstrom, Cole, and Trilling (reference 1); a linearized form of the Navier-Stokes equation in which the transverse viscous effects and the longitudinal viscous effects can be superposed was integrated. The present paper considers a source or sink flow of a viscous, heat-conducting, compressible fluid. This problem is a nonlinear flow case in more than one dimension for which the viscous effects are solely longitudinal. The relative simplicity of source flow is due to the fact that although the flow is two- or three-dimensional, the flow can be represented by single-parameter equations because of its cylindrical or spherical symmetry and, thus, it has no place for transverse viscous effects. The present solution may be considered a generalization of one-dimensional, viscous, heat-conducting shock flow to two or three dimensions; in such cases, expansion flows as well as compression flows are possible.

In the solution of the Navier-Stokes equation given herein, use is made of the second viscosity coefficient. The fundamental developments concerned with the second viscosity coefficient are given in Busemann's "Gasdynamik" (reference 2) published in 1931, derived by means of a gas-dynamic approach, and in Tisza's paper (reference 3), published in 1942, where the derivation is for ultrasonics and its implications for gas dynamics is only briefly mentioned. Since the development given by Busemann is very brief and Tisza's development is derived with a different approach, a more detailed account of the second viscosity coefficient in gas dynamics is given in the present paper.

SYMBOLS

x distance from source (or sink)

y,z space coordinates normal to x

u velocity in direction of source (or sink)

v,w velocities normal to u

$\overline{V}$ velocity vector

$\overline{u} = \dfrac{u}{u_{max}}$

τ viscous stress

π total stress

F surface of sphere or circumference of circle; also, cross-sectional area of tube with slightly varying cross section

a	speed of sound
ρ	density
t	time
T	absolute temperature
p	pressure
M	Mach number (u/a)
s	entropy
h	enthalpy
R	gas constant
μ	first viscosity coefficient
μ'	second viscosity coefficient
k	heat conductivity coefficient
c	mean molecular velocity
l	mean free path
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
γ	ratio of specific heats (c_p/c_v)
C_1	constant

$$C_2 = \frac{\mu_o}{\rho u x}$$

$$\phi' = \frac{d \log x}{d \log \overline{u}}$$

Subscripts:

o	stagnation conditions
max	maximum

A bar above a quantity denotes a vector except in the case of $\overline{u}$ where $\overline{u}$ represents u/u_{max}.

FUNDAMENTAL EQUATIONS AND THEIR INTEGRATION

The general flow equations are (references 2 and 4):

The continuity equation is

$$\operatorname{div}(\rho\bar{v}) = 0 \tag{1}$$

The equation of motion is

$$\rho \frac{D\bar{v}}{dt} = -\operatorname{div} \pi \tag{2}$$

where π is the total stress tensor which has the following relation to the pressure p and the viscous stress tensor τ:

$$\begin{pmatrix} \pi_{xx} & \pi_{xy} & \pi_{xz} \\ \pi_{yx} & \pi_{yy} & \pi_{yz} \\ \pi_{zx} & \pi_{zy} & \pi_{zz} \end{pmatrix} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

The energy equation is conveniently written in the form

$$\operatorname{div}\left[\rho\bar{v}\left(\frac{\bar{v}^2}{2} + h\right) - \left(k \operatorname{grad} T - \bar{v}.\tau\right)\right] = 0 \tag{3}$$

where $k \operatorname{grad} T$ represents the heat flow, $\bar{v}\cdot\tau$ represents the flow of work of the stress tensor τ, and the quantity h is the enthalpy of the flow. The inner or contracted multiplication $\bar{v}\cdot\tau$ (or $v_i\tau_{il}$) (for example, reference 5) yields a vector with the components

$$(\bar{v}\cdot\tau)_x = u\tau_{xx} + v\tau_{yx} + w\tau_{zx}$$

$$(\bar{v}\cdot\tau)_y = u\tau_{xy} + v\tau_{yy} + w\tau_{zy}$$

$$(\bar{v}\cdot\tau)_z = u\tau_{xz} + v\tau_{yz} + w\tau_{zz}$$

The expression $\operatorname{div}(\overline{V}\cdot\tau)$ can be written in the form

$$\frac{\partial}{\partial x}\left(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}\right) + \frac{\partial}{\partial y}\left(u\tau_{yx} + v\tau_{yy} + w\tau_{yz}\right) +$$

$$\frac{\partial}{\partial z}\left(u\tau_{zx} + v\tau_{zy} + w\tau_{zz}\right)$$

This expression is of the same form as the equation given in reference 4 with the exception that in the case of reference 4 the stresses include the pressure term.

Due to the symmetry of source flow, instead of transforming equations (1), (2), and (3) into polar form, x and u can be introduced for the polar coordinate and polar velocity, respectively. The continuity equation becomes

$$\rho uF = \text{Constant} = C_3 \tag{4}$$

In order to express the equations of motion and of energy for source flow, the components of the stress tensor are needed (references 2 and 4) and are

$$\tau_{xx} = -2\mu \frac{\partial u}{\partial x} - \frac{2}{3}(\mu' - \mu)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\tau_{yx} = -\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{zx} = -\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

With the use of the symmetry relations for source flow

$$\frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = \frac{u}{x}$$

and conditions for irrotationality, the equation of motion becomes for three-dimensional source flow

$$\rho u \frac{du}{dx} = -\frac{dp}{dx} + \frac{d}{dx}\left[2\mu \frac{du}{dx} + \frac{2}{3}(\mu' - \mu)\left(\frac{du}{dx} + 2\frac{u}{x}\right)\right] + 4\mu \frac{d}{dx}\left(\frac{u}{x}\right) \tag{5}$$

and for two-dimensional source flow

$$\rho u \frac{du}{dx} = -\frac{dp}{dx} + \frac{d}{dx}\left[2\mu \frac{du}{dx} + \frac{2}{3}(\mu' - \mu)\left(\frac{du}{dx} + \frac{u}{x}\right)\right] + 2\mu \frac{d}{dx}\left(\frac{u}{x}\right) \tag{6}$$

The equation of energy for three-dimensional source flow is

$$C_3 \frac{d}{dx}\left(\frac{u^2}{2} + h\right) - \frac{d}{dx} F\left[\frac{k}{c_p}\left(\frac{dh}{dx} + 2Pr \frac{d\frac{u^2}{2}}{dx}\right) + \frac{2}{3}(\mu' - \mu)\left(\frac{d\frac{u^2}{2}}{dx} + 2\frac{u^2}{x}\right)\right] = 0 \tag{7}$$

For two-dimensional source flow the term u^2/x replaces $2\frac{u^2}{x}$ in the energy equation. The Prandtl number Pr is given by $\mu c_p/k$.

By putting $\mu' = \mu$ and $Pr = \frac{1}{2}$ in equation (7), the simple case of constant total flow energy is obtained; that is,

$$\frac{u^2}{2} + h = \text{Constant}$$

A detailed discussion of the relation between μ' and μ is given in appendix A. In terms of the energy equation in the general form (equation (3)), the constancy of total flow energy is given by

$$k \text{ grad } T - \overline{v}\cdot\tau = 0$$

The compensation of heat flow and work flow is, of course, in line with the statement that no heat is added to the flow. The differences between source flow with constant total flow energy and with heat addition are discussed in appendix B.

With the use of the requirement of $\mu' = \mu$ for constant total energy flow, the equation of motion for three-dimensional source flow is

$$\rho u \frac{du}{dx} = -\frac{dp}{dx} + \frac{d}{dx}\left(2\mu \frac{du}{dx}\right) + 4\mu \frac{d}{dx}\left(\frac{u}{x}\right) \tag{8}$$

and for two-dimensional source flow

$$\rho u \frac{du}{dx} = -\frac{dp}{dx} - \frac{d}{dx}\left(2\mu \frac{du}{dx}\right) + 2\mu \frac{d}{dx}\left(\frac{u}{x}\right) \tag{9}$$

In terms of the area F, which for two- and three-dimensional flow is given by $2\pi x$ and $4\pi x^2$, respectively, equations (8) and (9) become identical:

$$\frac{dp}{dx} - \frac{d}{dx}\left(2\mu \frac{du}{dx}\right) - 2\mu \frac{d}{dx}\left(u \frac{d \log F}{dx}\right) = -\rho u \frac{du}{dx} \tag{10}$$

Now, the following substitutions are made in equations (8) and (9):

$$p = \frac{\rho a^2}{\gamma}$$

or

$$p = \rho \frac{\gamma - 1}{2\gamma} u_{max}{}^2(1 - \bar{u}^2)$$

where $a^2 = \frac{\gamma - 1}{2} u_{max}{}^2(1 - \bar{u}^2)$ and u_{max} is used as the reference value. With the assumption that μ varies in the form

$$\frac{\mu}{\mu_o} = \sqrt{\frac{T}{T_o}}$$

and with the substitution

$$T = \frac{\gamma - 1}{2\gamma R} u_{max}{}^2(1 - \bar{u}^2)$$

the following equation results:

$$\mu = u_{max} C_1 \sqrt{1 - \bar{u}^2}$$

where

$$C_1 = \frac{\mu_0}{\sqrt{T_0}} \sqrt{\frac{\gamma - 1}{2\gamma R}}$$

Equations (8) and (9) are now transformed to make the coordinate x nondimensional by first writing $\frac{d^2\bar{u}}{dx^2}$ in the form

$$\frac{d^2\bar{u}}{dx^2} = \frac{1}{x^2} \frac{d^2\bar{u}}{d \log x^2} - \frac{1}{x} \frac{d\bar{u}}{dx}$$

obtained through development of

$$\frac{d^2\bar{u}}{d \log x^2} = x \frac{d}{dx}\left(x \frac{d\bar{u}}{dx}\right) = x \frac{d\bar{u}}{dx} + x^2 \frac{d^2\bar{u}}{dx^2}$$

For the transformation from x to $\log x$, multiply equation (8) and (9) by x/ρ; the following equations result:

For cone flow

$$\frac{\gamma - 1}{2\gamma} \frac{d \log \rho}{d \log x}\left(1 - \bar{u}^2\right) + \frac{1}{\gamma} \bar{u} \frac{d\bar{u}}{d \log x} + 2C_2\bar{u}^2(1 - \bar{u}^2)^{-\frac{1}{2}}\left(\frac{d\bar{u}}{d \log x}\right)^2 +$$

$$4C_2\bar{u}^2\left(1 - \bar{u}^2\right)^{\frac{1}{2}} - 2C_2\bar{u}(1 - \bar{u}^2)^{\frac{1}{2}} \frac{d^2\bar{u}}{d \log x^2} - 2C_2\bar{u}(1 - \bar{u}^2)^{\frac{1}{2}} \frac{d\bar{u}}{d \log x} = 0 \tag{11}$$

and for wedge flow

$$\frac{\gamma - 1}{2\gamma} \frac{d \log \rho}{d \log x}(1 - \bar{u}^2) + \frac{1}{\gamma} \bar{u} \frac{d\bar{u}}{d \log x} + 2C_2\bar{u}^2(1 - \bar{u}^2)^{-\frac{1}{2}}\left(\frac{d\bar{u}}{d \log x}\right)^2 +$$

$$2C_2\bar{u}^2(1 - \bar{u}^2)^{\frac{1}{2}} - 2C_2\bar{u}(1 - \bar{u}^2)^{\frac{1}{2}} \frac{d^2\bar{u}}{d \log x^2} = 0 \tag{12}$$

The quantity C_2 is given by

$$C_2 = \frac{C_1}{\rho\bar{u}x} = \frac{\mu_o}{\sqrt{\gamma RT_o}}\sqrt{\frac{\gamma - 1}{2}}\,\frac{1}{\rho\bar{u}x} = \frac{\mu_o}{a_o}\sqrt{\frac{\gamma - 1}{2}}\,\frac{1}{\rho\bar{u}x}$$

Since $\bar{u}$ is actually a short notation for u/u_{max} where u_{max} was assumed to be the reference value and since

$$\frac{u_{max}}{a_o} = \sqrt{\frac{2}{\gamma - 1}}$$

C_2 may be written in the form

$$C_2 = \frac{\mu_o}{\rho ux}\,\frac{u_{max}}{a_o}\sqrt{\frac{\gamma - 1}{2}} = \frac{\mu_o}{\rho ux}$$

Now, the mass flow for two- and three-dimensional source or sink flow (wedge or cone flow) is given by

$$\rho uF = \rho ux2\pi$$

and

$$\rho uF = \rho ux^2 4\pi$$

For wedge flow the dimensionless parameter C_2, which has the form of an inverse Reynolds number, is a constant; whereas for cone flow

$$C_2 = \frac{\mu_o}{\rho ux} = \frac{\mu_o}{\rho ux^2}\,x = \text{Constant }(x)$$

or C_2 is proportional to x, the distance from the source or sink. For three-dimensional flow no advantage has been obtained by the introduction of $\log x$ since a dependency on x is hidden in C_2. In the present paper, the simpler case of wedge flow is discussed, since its simplicity permits a quick grasp of the fundamental nature of the flow problem.

Equation (12) can be simplified by introducing the continuity equation in the form

$$\frac{d \log \rho}{d \log x} = -\frac{1}{\bar{u}} \frac{d\bar{u}}{d \log x} - 1$$

Since in the equation the independent variable $\log x$ appears in much simpler form than the dependent variable $\bar{u}$, equation (12) can be further simplified by an interchange of the dependent and independent variables. This interchange is achieved by multiplying equation (12) by $\left(\frac{d \log x}{d\bar{u}}\right)^3$ and expressing $\frac{d^2\bar{u}}{d \log x^2}$ by

$$\frac{d^2\bar{u}}{d \log x^2} = \frac{d}{d \log x}\left(\frac{d\bar{u}}{d \log x}\right) = \frac{d}{d\bar{u}}\left[\left(\frac{d \log x}{d\bar{u}}\right)^{-1}\right]\frac{d\bar{u}}{d \log x}$$

$$= -\frac{d^2 \log x}{d\bar{u}^2}\left(\frac{d\bar{u}}{d \log x}\right)^3$$

Therefore,

$$\xi'^3\left[-\frac{\gamma - 1}{2\gamma}(1 - \bar{u}^2) + 2C_2\bar{u}^2(1 - \bar{u}^2)^{\frac{1}{2}}\right] + \xi'^2\left[-\frac{\gamma - 1}{2\gamma}(1 - \bar{u}^2)\frac{1}{\bar{u}} + \frac{1}{\gamma}\bar{u}\right] +$$

$$\xi' 2C_2\bar{u}^2(1 - \bar{u}^2)^{-\frac{1}{2}} + \xi'' 2C_2\bar{u}(1 - \bar{u}^2)^{\frac{1}{2}} = 0 \tag{13}$$

or

$$\xi'' = -\frac{\bar{u}}{1 - \bar{u}^2}\xi' - \frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{\frac{4\gamma C_2}{\gamma - 1}\bar{u}^2\sqrt{1 - \bar{u}^2}}\xi'^2 + \left(\frac{\gamma - 1}{4\gamma C_2}\frac{\sqrt{1 - \bar{u}^2}}{\bar{u}} - \bar{u}\right)\xi'^3 \tag{14}$$

where

$$\xi' = \frac{d \log x}{d\bar{u}}$$

To avoid odd powers in the coefficients for convenience, reduce equation (14) by introduction of a new variable

$$\phi' \equiv \frac{d\phi(\log \bar{u})}{d \log \bar{u}} \equiv \frac{d \log x}{d \log \bar{u}}$$

such that it becomes

$$\phi'' = \phi'\left(1 - \frac{\bar{u}^2}{1 - \bar{u}^2}\right) - \phi'^2 \frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{\frac{4\gamma C_2}{\gamma - 1}\bar{u}^2\sqrt{1 - \bar{u}^2}} + \phi'^3\left(\frac{\gamma - 1}{4\gamma C_2}\frac{\sqrt{1 - \bar{u}^2}}{\bar{u}^2} - 1\right) \qquad (15)$$

In the present form, $\bar{u}$ is always raised to the second power and, thus, since the second power term is always positive, the sign of C_2 will determine whether the flow will pass in the positive or negative direction through a given wedge. Specifically, since

$$C_2 = \frac{C_1}{\rho\bar{u}x} = \frac{\mu_o}{\rho u x}$$

C_2 will be positive if u and x are of the same sign and negative if they are of opposite sign. Since, as previously mentioned, x is the distance from the apex of the wedge, a positive C_2 signifies a flow through a diverging wedge source, whereas a negative C_2 indicates a flow through a converging wedge sink.

The physical significance of C_2 may be seen by making the substitutions (see, for example, reference 6)

$$\mu = 0.499\rho c l$$

and

$$c = \sqrt{\frac{8}{\pi\gamma}}\, a$$

and by applying relations previously used. The relation for C_2 becomes

$$C_2 = K_1 \frac{l}{x} \frac{a_o}{u}$$

where

$$K_1 = 0.499\sqrt{\frac{8}{\pi\gamma}}$$

As may be seen from flow tables (for example, reference 7), the ratio a_o/u will vary comparatively little for a large range of supersonic and even subsonic Mach numbers. For the supersonic Mach number range, a_o/u will be represented to a good approximation by a constant and C_2 will be proportional to l/x with a proportionality constant not far from one. The significance of C_2, a reciprocal Reynolds number, may perhaps be more conveniently seen by thinking of x as the width of a given wedge section through which the flow passes. Since the density is inversely proportional to the mean free path (reference 6), C_2 is about inversely proportional to both the density of the flow and the width of flow section. If $C_2 = 1$ is assumed as a criterion for the case where viscous effects due to compressibility are of importance, the isolated longitudinal viscous effects will be negligible for hypersonic tunnels operating in the range of atmospheric stagnation conditions even though the minimum section of the tunnel may be small based on engineering standards. If the minimum section of the hypersonic tunnel is to influence greatly the isolated longitudinal viscous effects without becoming impracticably small, the mean free path has to be increased to the order of the width of the minimum section; that is, the stagnation density of the flow has to be correspondingly small. For three-dimensional flow C_2 increases with the distance from the source and, thus, the viscous effects will be correspondingly influenced.

Since the nature of equation (15) rules out a closed integration, the integration is performed numerically, which is done conveniently by a modified isocline method. The modification consists in requiring $\frac{\phi''}{\phi'} = \text{Constant}$ along the modified isoclines in the $\phi',\bar{u}$ plane instead of requiring $\phi'' = \text{Constant}$ as would be necessary for true isoclines; ϕ''/ϕ' represents reciprocal subtangents. The equation of the modified isoclines is given by

$$\phi' = \frac{1}{2}\left\{\frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{(1 - \bar{u}^2) - C_2\frac{4\gamma}{\gamma - 1}\bar{u}^2\sqrt{1 - \bar{u}^2}} \pm \sqrt{\left[\frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{(1 - \bar{u}^2) - C_2\frac{4\gamma}{\gamma - 1}\bar{u}^2\sqrt{1 - \bar{u}^2}}\right]^2 - \frac{4(1 - 2\bar{u}^2)}{(1 - \bar{u}^2)\left(\frac{\gamma - 1}{4\gamma}\frac{\sqrt{1 - \bar{u}^2}}{C_2\bar{u}^2} - 1\right)} + \frac{\phi''}{\phi'}\frac{4}{\frac{\gamma - 1}{4\gamma C_2}\frac{\sqrt{1 - \bar{u}^2}}{\bar{u}^2} - 1}}\right\} \tag{16}$$

The integration curves in the $\phi',\bar{u}$ plane are obtained by connecting the slopes on the isoclines corresponding to various values of ϕ''/ϕ' (see figs. 1 and 2). The integration curves in $\phi',\bar{u}$ plane are most conveniently integrated in the $\bar{u},x$ plane step by step by means of the expression

$$\Delta\phi = \phi' \, \Delta \log \bar{u}$$

Since the curves $\phi(\log \bar{u})$ represent an integration of the curves $\phi'(\log \bar{u})$, they are determined except for an arbitrary constant. The arbitrary constant means that the curves $\phi(\log \bar{u})$ in figure 3 may be shifted along the x-axis; in other words, the two-parameter family of curves obtained from the second-order differential equation has been converted into a single-parameter family and a translation family. This shifting of the solutions along the x-axis could have been predicted from the fact that for two-dimensional flow C_2 is independent of the distance from the source; for the simple isentropic case, of course, the solutions can also be shifted. It can be expected that for three-dimensional source flow, a simple introduction of a translation family would no longer be possible; thus, a second-order differential equation may have to be solved numerically instead of reducing the problem to the much simpler solution of a first-order differential equation.

The integration for each particular (integral) curve was performed by starting at some point of the flow field and following the structure of the field from there on. This step-by-step approach has the aspect of an initial-value problem. Since the integral curves in the physical plane can be shifted along the x-axis, the curves can be placed in a position such that they can be compared with the integral curves for the case where the viscous effects due to source flow are zero. In the numerical integration a choice had to be made for the magnitude of the parameter C_2. The values of $C_2 = \pm 0.1$ were selected, since trial computations for various values of C_2 indicated that in this case the longitudinal viscous effects due to compressibility would be comparatively small. Such solutions should be of special interest since they can be expected to have certain aspects in common with both the cases of zero viscous effects $C_2 = 0$ and finite viscous effects. In order to understand fully the contribution of the various aspects, the structure of equations (15) or (16) is analyzed for the neighborhood of $C_2 = 0$.

DISCUSSION OF SOLUTIONS

The Neighborhood of $C_2 = 0$ with a Discussion of $C_2 = 0$

To investigate the neighborhood of $C_2 = 0$, equation (15) is multiplied by C_2; then C_2 appears as a coefficient of ϕ'', the highest-order term of the differential equation. The problem becomes one of deciding what happens when the coefficient of the highest-order term of the differential equation approaches 0. A similar problem arises in connection with the development of the boundary-layer equations and equations of shocks in constant cross section where μ (instead of C_2) approaches 0 (references 1, 8, and 9). (The term shock is used herein for compressions with finite μ and k as well as for compressions with $\mu \longrightarrow 0$ and $k \longrightarrow 0$.) Generally, for this type of problem, the effects of viscosity and heat conduction are of such a nature that they smooth out the discontinuities existent when the coefficient of the highest-order term of the differential equation is set equal to 0. This smoothing of the discontinuities applies, of course, only to the immediate neighborhood of $C_2 = 0$. For the present case, when $C_2 = \pm 0.1$, values which are finite but small, such a process will be true only approximately, but it is very useful in understanding the flow structure.

$C_2 = 0$.- The discontinuities for $C_2 = 0$, although obtainable from physically plausible considerations, are obtained from equation (15). Equation (15) is rewritten after multiplying it by C_2

$$C_2\phi'' = C_2\phi'\left(1 - \frac{\bar{u}^2}{1 - \bar{u}^2}\right) - \phi'^2 \frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{\frac{4\gamma}{\gamma - 1}\bar{u}^2\sqrt{1 - \bar{u}^2}} +$$

$$\phi'^3 \frac{\gamma - 1}{4\gamma}\frac{\sqrt{1 - \bar{u}^2}}{\bar{u}^2} - C_2\phi'^3 = 0 \tag{17}$$

If C_2 is set equal to 0 and ϕ'' remains finite and arbitrary, equation (17) reduces to

$$\phi'^2 \frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{\frac{4\gamma}{\gamma - 1}\bar{u}^2\sqrt{1 - \bar{u}^2}} - \phi'^3 \frac{\gamma - 1}{4\gamma}\frac{\sqrt{1 - \bar{u}^2}}{\bar{u}^2} = 0 \tag{18}$$

This equation has one double root

$$\phi'_{I,II} = 0$$

and a third root

$$\phi' = \frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{1 - \bar{u}^2}$$

which is the equation for the isentropic curve. The isentrope and the $\phi' = 0$ axis will be the loci of finite and arbitrary values of ϕ''. This fact is illustrated in figure 4(a) representing the $\phi',\bar{u}$ plane by fans of lines distributed on these curves. The solutions of $\phi' = 0$, or the equivalent of $\frac{d \log \bar{u}}{d \log x} = \infty$, actually represent the previously mentioned discontinuities existent when the coefficient of the highest-order term of the differential equation is set equal to 0. For the smoothing of the discontinuities by viscosity, the highest-order term of the differential equation has to be kept finite which can be accomplished by making ϕ'' equal to ∞ (see dashed lines in fig. 4(a)). Then, since $C_2\phi''$ is indeterminate ϕ' can assume any value to compensate for the indeterminate term. Because ϕ' is arbitrary the connections between the isentrope and the discontinuities $\phi' = 0$ can be established at every point of the isentrope. These connections represent the smoothing effects in their embryonic stage. The letters in figure 4 are given to correlate the (a) and (b) parts.

Since the discontinuities do not follow the isentropic law, entropy variations occur. (The entropy balance is given in appendix C.) The entropy variations for the discontinuities can be understood from the fact that the viscous-stress terms and the heat-conduction terms (since the total flow energy is constant) are essentially given by

$$\mu \frac{du}{dx} \approx C_2 \rho u x \frac{du}{dx}$$

For given values of ρ, u, and du the viscous effects are constant if C_2 and dx/x decrease proportionately down to 0. For an infinite value of x (from a source of infinite strength), $C_2 = 0$ means flow through a constant cross section; the reduction of the source-flow equations to shock flow in a constant cross section is given in appendix D.

Since the velocity of source flow depends only on the distance from the source, flow discontinuities may also arise because of the termination of source flow by an adjoining non-source flow. Depending on whether the cross-sectional variation at the junction of source flow and non-source flow is discontinuous in the first derivative or is discontinuous itself, the terminating flow discontinuities are represented by $\frac{du}{dx} = \infty$ or by velocity jumps. Naturally the source flow does not have to be terminated by discontinuities in the cross-sectional variation at the junction or its first derivative; rather, termination of source flows is possible for discontinuities in any derivative. At the junction of non-source-flow boundaries with source-flow boundaries, the symmetry of source flow no longer holds. Since the present calculations deal with pure source flow, they neglect the disturbance of the flow symmetry due to the termination of source flow. Such unsymmetrical effects will tend to disappear for slightly diverging source flows (or slightly converging sink flows) since for them the flow velocity tends to be constant in a given flow cross section.

The discontinuity $\frac{du}{dx} = \infty$ at sonic velocity is due to the fact that a nonviscous source or sink flow is also terminated or limited as it cannot pass beyond the location where the sonic velocity occurs. For two-dimensional source flow the so-called "limiting line" (reference 10) of the flow is a circle at sonic velocity. Because of the use of symmetrical conditions of source flow, in the present case the limiting line appears as a point of the isentrope at $M = 1$ in the physical plane $(\bar{u},x)$.

<u>The immediate neighborhood of $C_2 = 0$.</u>- In investigating the immediate neighborhood of $C_2 = 0$ (see figs. 5 and 6), the purpose is not to give a detailed mathematical account of the "neighborhood problem" but rather to inspect equation (15) for certain features which should give an insight into its solutions without requiring actual integration. Since C_2 is now finite, though in the immediate neighborhood of 0, the right side of equation (17) will no longer be 0 for finite and arbitrary values of ϕ'' and thus some of the simplicity inherent in the solution for $C_2 = 0$ will be lost. Equation (15) indicates, however, that for finite values of C_2 the equality of 0 of its right side (which also results in $\phi'' = 0$) will give certain useful relations. Namely, the $\phi'' = 0$ axis will not only be a locus of constant slopes $\frac{d\phi'}{d \log \bar{u}} = \phi''$ but also an integration curve (in the $\phi',\bar{u}$ plane) since its own slope is $\frac{d\phi'}{d \log \bar{u}} = 0$. Since the integration curves of equation (15) cannot cross each other, as the coefficients of equation (15) are single-valued functions of $\bar{u}$, the fact that the integration curve $\phi' = 0$ is a straight line can be used for orientation among the infinite group of integration curves in the $\phi',\bar{u}$ plane. Through inspection of equation (15), orientation is also possible in the direction field given by the curves along which $\frac{d\phi'}{d \log \bar{u}} = \phi'' = \text{Constant}$ (in the $\phi',\bar{u}$ plane). Since the direction field given by equation (15) is continuous, each curve of the system of all curves of $\phi'' = \text{Constant}$ furnishes a dividing line for the slopes ϕ'' in the plane $\phi',\bar{u}$. Similar to what was done previously for the integration curves, one particular dividing line for the slopes is chosen for orientation among the lines of constant slope ϕ'' in the $\phi',\bar{u}$ plane. The dividing line $\phi'' = 0$ is chosen since in that case equation (15) yields as one solution the $\phi' = 0$ axis. The general trend of the other two solutions of equation (15) for $\phi'' = 0$ $\left(\text{or of the slightly modified equation (16) for } \frac{\phi''}{\phi'} = 0\right)$ for the immediate neighborhood of $C_2 = 0$ is taken from figures 1 and 2 where the $\frac{\phi''}{\phi'} = 0$ curves are given for $C_2 = \pm 0.1$. Further, for orientation in the direction

field, the fact contained in equation (15) can be used that as ϕ' tends to infinity, ϕ'' (the slope in the $\phi',\bar{u}$ plane) will do the same. The only exception is that when the coefficient of the term ϕ'^3, which has the highest degree in equation (15), becomes 0, then ϕ'' will be indefinite for an infinite ϕ'.

The direction fields in the $\phi',\bar{u}$ plane for the positive neighborhood $C_2 = +0$ (fig. 5) and the negative neighborhood $C_2 = -0$ (fig. 6) of $C_2 = 0$ may now be constructed. In order to obtain a clear illustration, the size of the regions of finite slopes (furnishing the smoothing effects) has been exaggerated in figures 5 and 6. The following features in the neighborhood solutions may be singled out. For the immediate neighborhood of $C_2 = 0$ integration curves exist which essentially remain in the neighborhood of the isentropic curve for $C_2 = 0$. The direction fields also indicate that integration curves due to compression shocks or due to flow changes caused by termination of source or sink flow will practically join the neighborhood solutions of the isentropic curves. In figures 7 and 8 a summary of the behavior of the integral curves for the immediate neighborhood of $C_2 = 0$ is made. The arrows in these figures indicate the velocity increase and decrease corresponding to source $(C_2 = +0)$ and sink $(C_2 = -0)$ flow, respectively.

The curves of figures 5 to 8 are numbered (as far as possible) in accordance with the comparative curves in figures 1, 2, and 3. Curves 1, 2, and 3 apply to the case of $C_2 = +0$ or source flow. Curve 1 represents an expansion for infinite source flow and is essentially due to the smoothing by viscosity between the discontinuity limiting the flow at sonic velocity and the supersonic branch of the isentrope. Curve 2 represents a compression shock in infinite source flow. Behind the shock, subsonic velocity exists which may be decelerated to zero in the diverging source flow. Curve 3 represents the smoothing by viscosity of expansion flow due to termination of source flow. The curves 4, 5, and 6 represent corresponding effects for $C_2 = -0$ or sink flow. The curves are discussed in greater detail for flows with $C_2 = \pm 0.1$, presented in figures 1, 2, and 3.

Now, a comparison can be made between the flow structures for the immediate neighborhood of $C_2 = 0$ and for $C_2 = 0$. For the immediate neighborhood C_2 is finite, though small, and thus on a strict mathematical basis the direction fields of the differential equation are such that all the solutions are represented by separate curves. Figures 5 and 6 for the direction fields of the immediate neighborhood, however, indicate that solutions due to shocks or termination of source-or-sink flow boundaries will join for all practical purposes the neighborhood solutions of the isentropes. For $C_2 = 0$, the junctions between discontinuous changes and the isentrope are correct on a strict mathematical basis. For all practical purposes the neighborhood case still has certain aspects which were exact for the case of $C_2 = 0$. The solutions for $C_2 = \pm 0.1$ given in figures 1, 2, and 3 indicate that the viscous smoothing effect of discontinuities, due to termination of source or sink flow or due to shocks ($\mu \longrightarrow 0$, $k \longrightarrow 0$) in infinite source or sink flow (curves 2, 3, 5, 6, and 7), will approach the expansion curves 1 and 4 for infinite source or sink flow with sufficient rapidity so that they may be joined with them for all practical purposes. In other words, the plotted solutions for the finite values $C_2 = \pm 0.1$ are close enough to those for $C_2 = 0$ to show the typical aspects of the immediate neighborhood solutions $C_2 = +0$ and $C_2 = -0$.

The Case of $C_2 = \pm 0.1$

The numerical integration, as previously stated, was performed by starting at some point of the flow field and by following the structure of the field from there on. In integrating this initial-value problem, only two curves in figures 1, 2, and 3, curves 1 and 4 (which may be shifted), exist which extend from the neighborhood of the source or sink to infinity without being interrupted by curves rapidly deviating essentially in an exponential manner. For finite values of C_2 the rapidly deviating curves have to be actually separate curves from 1 and 4, but for the small values of $C_2 = \pm 0.1$ a junction will still be correct for all practical purposes. Since the exponentially deviating curves approximately join the more gradual curves 1 and 4, integration is advisable in the direction opposite to that in which the exponential deviations are expected to occur. Otherwise, difficulties will be encountered in fitting these solutions into one particular exponential deviation (with certain initial values ϕ and ϕ') of the gradual solutions as exhibited by curves 1 and 4. Similar difficulties were found in the analysis of reference 11, which deals with one-dimensional shock flow for arbitrary Prandtl numbers; other fundamental aspects of that flow structure are presented in a very clear manner in reference 12.

Curve 1 in figures 1 and 3 is the equivalent of curve 1 in figure 5. It thus essentially represents the viscous counterpart of isentropic expansion flow for the case of an infinite diverging wedge. Curve 1 was shifted along the x-axis into such a position as to make possible a comparison with isentropic flow through the same area ratio. (It should be noted that $\log x_2 - \log x_1 = \log \frac{x_2}{x_1} = \log \frac{F_2}{F_1}$.)

Curve 2 represents a compression shock for infinite source flow; its velocity distribution is essentially the same as in the well-known shock in constant cross section. Curve 3 represents an expansion due to termination of source flow. Curve 4 represents the expansion flow through an infinite converging wedge. The flow starts at zero velocity and goes through the subsonic range and then essentially smooths out the region which is abruptly limited at sonic velocity for isentropic flow. The surprising thing about this expansion flow is that it penetrates beyond the sonic section or the minimum section for isentropic flow. The meaning of this penetration is discussed subsequently. Curve 5 represents a compression shock in converging supersonic infinite sink flow. Such a shock is actually unstable, but such considerations do not enter into the present problem. Curve 7 represents another compression shock for infinite sink flow which is unusual in that it is entirely in the supersonic region. Its existence is possible because expansion curve 4, for converging-wedge flow, penetrates into the supersonic region. Finally, curve 6 represents an expansion due to termination of sink flow in the subsonic-flow region. Shock 7 follows more closely the trend of supersonic isentropic compression than does shock 5. Curves which will follow this trend even more closely could have been given. In connection with the possibility of existence of such various compression curves, it should be noted that for nonviscous flow through converging wedges, solutions do exist with shocks and without shocks.

Since the structure of the flow field has been treated, certain general aspects can now be discussed. For example, in figure 3 the curves representing viscous flow solutions have a smaller slope than the corresponding curves for $C_2 = 0$. This smaller slope is immediately apparent for curves which for $C_2 = 0$ are represented by discontinuities $\frac{d\bar{u}}{dx} = \infty$ or lines of discontinuities. Such is the case for the shocks, curves 2 and 5, and for source-flow terminations with the appropriate boundary changes. (See curves 3 and 6.) In order to indicate the fact that curves 1 and 4 have a smaller slope than their isentropic $\left(C_2 = 0\right)$ counterpart, the isentrope has to be shifted along the x-axis. The smaller slope of the viscous-flow curves may also be expressed by the statement that the viscous flow requires a larger area ratio to go through a given velocity change than does the corresponding flow for $C_2 = 0$. This behavior could have been more or less expected except for the case of viscous sink flow converging to the section where $M = 1$, given by curve 4. Namely, viscous effects alone are known (for example, reference 2) to cause a given mass flow approaching $M = 1$ to require a larger cross section than the same isentropic mass flow. In other words, for viscous effects alone the flow approaching $M = 1$ can only pass through smaller area ratios than the same isentropic mass flow. The reason for the fact that the flow given by curve 4 requires a larger area ratio is based on the fact that heat can be conducted downstream in the present case. In summary, this behavior means that the familiar heat-conduction effects combined with the peculiar longitudinal viscous effects alone (no transverse viscous effects) may cause a given mass flow to pass through a cross section smaller than the isentropic minimum section; in other words, a larger mass flow may pass through a given isentropic minimum cross section than for isentropic flow. This problem is discussed in greater detail in a subsequent section and in appendix C.

Now, a few general aspects of the connection of the shocks and terminating solutions with the solutions for infinite source or sink flows, curves 1 and 4, can be discussed. Figures 1 to 3 show that for flow through diverging wedges, source flow, the exponential deviations from the expansion curve 1 become larger in the direction of increasing velocities (see curves 2 and 3). For converging-wedge flow, sink flow, the exponential deviations in the neighborhood of the expansion curve 4 increase with reduced velocities (see curves 5 and 6). Since both expansion flows are, of course, in the directions of increasing velocities, the effects of shocks and source or sink terminations will penetrate both upstream and downstream. The nature of the criterions for upstream and downstream penetration can be seen from the simple example of the shock in constant cross section; namely, since only shock from supersonic to subsonic velocities are possible, the effects of a compression shock will penetrate upstream on the supersonic side and downstream on the subsonic side. The same behavior is exhibited for the present case of source flow. For example, curves 2 and 3 indicate that the effects of shock or termination of source flow penetrate upstream in the supersonic region of expansion curve 1 of the source flow; whereas curves 5 and 6 indicate that the same effects penetrate downstream in the subsonic region of expansion curve 4 of the sink flow. The shocks show, of course, both types of penetration. Effects of shock 2 penetrate downstream and join the subsonic part of the isentrope. For the present cases of source or sink flow, exceptions exist to these criterions of subsonic and supersonic penetration. For example, shock 7 penetrates downstream into the supersonic velocity region of curve 4. This particular supersonic velocity region is characterized by the fact that curve 4 is continued to $\overline{u}$ values which would have been prohibited for the case that $C_2 = 0$. The behavior of the exponential penetration may thus be summarized. In the flow regions which were not prohibited in the case of $C_2 = 0$, flow changes (for example, like those of shocks in infinite wedge flow or due to termination of source or sink flow) will penetrate upstream in supersonic regions and downstream in subsonic regions. In regions prohibited for $C_2 = 0$, the nature of the penetration effect if given by the character of the flow of the corresponding expansion curve before it enters the region prohibited for the case of $C_2 = 0$.

Certain limitations of the solutions in figures 1, 2, and 3 exist which are of importance for the interpretation of these results. For example, curve 1, the expansion curve for infinite source flow, will not reach the maximum flow velocity but rather will be asymptotic to a velocity equal to about 0.7 of the maximum velocity. The existence of an asymptote for viscous expansion flow through the diverging cross section of source flow or the existence of the asymptote itself as a monotone solution could have been expected. However, the asymptote is reached for the case that the normal stress $\pi_{yy} = 0$, which is a condition for which the Navier-Stokes equation reaches its limit of applicability.

The Limits of Applicability of the Navier-Stokes Equation

Based on the kinetic theory of gases (see discussion of Burnett's work in reference 13 (especially p. 271)), the Maxwell distribution is regarded as a first approximation of the Boltzmann equation for the general distribution function and the viscous-stress terms of the Navier-Stokes equation are regarded as a second approximation. The limit of applicability of the Navier-Stokes equation is considered to be reached or passed when the terms of the third approximation reach the same order of magnitude as the terms of the second approximation in the Navier-Stokes equation. Actually, of course, no sharp limit exists for the applicability of the Navier-Stokes equation, but the reaching of the same order of second- and third-approximation terms is a sufficiently broad criterion or barrier to be generally accepted. A discussion of the limit of applicability of the Navier-Stokes equation is furthermore simplified by the fact that the ratio of the third-approximation terms and the second-approximation terms is of the same order as the ratio of the terms of the second approximation and the first approximation (see also reference 14, p. 453). For the present case, the limit of applicability of the Navier-Stokes equation is reached when the viscous stresses reach the order of magnitude of the fluid pressure.

Since the present solution is based on the condition $\mu' = \mu$, the normal stresses π_{xx} and π_{yy} are (references 2 and 4):

$$\pi_{xx} = p - 2\mu \frac{du}{dx}$$

and

$$\pi_{yy} = p - 2\mu\frac{u}{x}$$

Actually, a choice exists between two limits of applicability of the Navier-Stokes equation; one for

$$p = 2\mu\frac{u}{x}$$

and the other for

$$p = 2\mu\,\frac{du}{dx}$$

In terms of the variables used in the present paper, these two breakdown criterions are expressed by

$$\left(\frac{p}{2\mu\frac{u}{x}}\right)_{\text{Breakdown}} = \frac{4\gamma C_2}{\gamma - 1}\,\frac{\sqrt{1 - \bar{u}^2}}{\bar{u}^2} = 1$$

and

$$\left(\frac{p}{2\mu\,\frac{du}{dx}}\right)_{\text{Breakdown}} = \frac{4\gamma C_2}{\gamma - 1}\,\frac{\bar{u}^2}{\sqrt{1 - \bar{u}^2}}\,\frac{d\log x}{d\log\bar{u}} = 1$$

Since $\sqrt{1 - \bar{u}^2}$ is positive because it is related through the temperature T to the necessarily positive viscosity, the first breakdown criterion $p = 2\mu\frac{u}{x}$ has a meaning only for positive values of C_2, that is, for flow through a diverging wedge. For $C_2 = 0.1$ and $\gamma = 1.4$ (for air), $\bar{u} = 0.7095$. This value is the asymptotic value for which the gradient $\frac{d\bar{u}}{dx}$ is 0 for the expansion flow through an infinite diverging wedge (curve 1, figs. 1 to 3). Also, all expansion curves for diverging wedge flow will have to be cut off at $\bar{u} = 0.7095$. The condition $p = 2\mu\frac{u}{x}$ results also in $\pi_{yy} = 0$; thus, the stress normal to the streamlines is 0. The flow of the gas is thus no longer forced to follow the diverging streamlines, or, in other words, if the divergence of the wedge were to be increased, the flow would no longer follow.

Two possibilities, those of expansion and compression, exist for the second breakdown criterion $p = 2\mu\,\frac{du}{dx}$. In table I, the ratios of $\left(\frac{p}{2\mu\,\frac{du}{dx}}\right)_{\text{Plot}}$ to $\left(\frac{p}{2\mu\,\frac{du}{dx}}\right)_{\text{Breakdown}}$ are given for the various integration curves in figure 3. These ratios are equal to ratios of

$$\left(\frac{d \log \bar{u}}{d \log x}\right)_{\text{Breakdown}} \quad \text{to} \quad \left(\frac{d \log \bar{u}}{d \log x}\right)_{\text{Plot}} \quad \text{or} \quad \left(\frac{1}{\phi'}\right)_{\text{Plot}}$$

where ϕ' is the ordinate in figures 1 and 2.

Table I indicates, for example, that for compression shocks in figures 1, 2, and 3, breakdown of the Navier-Stokes equation will occur in certain flow regions within the shocks similarly to the well-known case of one-dimensional shocks. (It should be pointed out that the breakdown criterion $\frac{p}{2\mu \frac{du}{dx}} = 1$ is less severe for compressions than for expansions for which it coincides with $\pi_{xx} = 0$.) The table, furthermore, shows that for the expansion curve 3 based on source flow, breakdown due to $p = 2\mu\frac{u}{x}$ ($\bar{u} = 0.7095$) occurs before breakdown due to $p = 2\mu \frac{du}{dx}$. The table also indicates that the breakdown of the infinite source (curve 1) and sink (curve 4) solutions will occur in those flow regions in which, for isentropic flow, no solutions would have existed at all. Such behavior was already suggested in previous considerations.

As is indicated by $C_2 = \text{Constant } (x)$ for three-dimensional source flow with constant total energy, the limit of applicability of the Navier-Stokes equation will be reached sooner than for two-dimensional flow ($C_2 = \text{Constant}$) if x increases and later if x decreases. Since the equations for two- and three-dimensional flow differ essentially only by the dependence of C_2 on x, for flow changes in small regions (small changes in x), like those occurring in the regions of shocks and flow termination, only a small difference will exist between two- and three-dimensional flow.

Flow through a Curved Minimum Section Joined to a Sink Flow

Since a compressible source or sink flow cannot reach the source or sink, the problem arises concerning its continuation. The problem is of special interest in the case of the sink flow given by curve 4 which is able to penetrate beyond the $M = 1$ section. In this section flow in a flow filament through a curved minimum section joined to the sink flow is treated. Since the expansion flow of curve 4 passes to smaller cross sections than the isentropic minimum section, it would appear plausible on first thought that the flow would also pass through the curved minimum section. In a curved minimum section joined to the sink flow or converging wedge flow, the cross section is further reduced, however, below its value at the junction and, thus, a separate investigation has to be made by solving the differential equation for arbitrary boundary shapes.

Since the solution would result in a laborious numerical task, an attempt will be made to gain information by a study of the nature of the differential equation without actually solving it. For this purpose, the two-dimensional-flow equations will not be investigated, but equations simplified by assuming the flow to be quasi-one-dimensional with slightly variable cross section (filament flow) will be investigated instead. (The actual two-dimensional flow through the curved minimum passage, unlike the wedge flow, also contains transverse viscous effects. These effects are, however, eliminated for the simplified case of filament flow.) For this case, the flow equations may be written in the same form as for wedge flow, with the only difference that F is no longer equal to $2\pi x$ but that the variation of F as a function of x and the flow variables are arbitrary. The fundamental equations are transformed in such a manner that the effect of the flow variables on the variation in cross section may be easily recognized.

After the expression

$$\pi_{xx} = p - 2\mu \frac{du}{dx}$$

is substituted in equation (10), the equation of motion is obtained in the following form:

$$\rho u \frac{du}{dx} = 2\mu \frac{d}{dx}\left(u \frac{d \log F}{dx}\right) - \frac{d\pi_{xx}}{dx}$$

The continuity equation is

$$\rho u F = \text{Constant} = C_3$$

Also, for the present case of constant-energy flow, as previously determined,

$$T = \frac{\gamma - 1}{2\gamma R} u_{max}^2(1 - \bar{u}^2)$$

$$\mu = u_{max} C_1 \sqrt{1 - \bar{u}^2}$$

Now, the equation of motion may be rewritten as

$$-\rho\bar{u}\,\frac{d\bar{u}}{dx} + 2C_1\sqrt{1 - \bar{u}^2}\,\bar{u}\,\frac{d^2\log F}{dx^2} + 2C_1\sqrt{1 - \bar{u}^2}\,\frac{d\bar{u}}{dx}\,\frac{d\log F}{dx} - \frac{1}{{u_{max}}^2}\,\frac{d\pi_{xx}}{dx} = 0$$

or

$$\frac{d^2\log F}{dx^2} = \left(\frac{\bar{C}_3}{2C_1\sqrt{1 - \bar{u}^2}} - \frac{dF}{dx}\right)\frac{1}{\bar{u}F}\,\frac{d\bar{u}}{dx} + \frac{1}{2C_1\sqrt{1 - \bar{u}^2}\,\bar{u}}\,\frac{1}{{u_{max}}^2}\,\frac{d\pi_{xx}}{dx} = 0$$

where

$$\bar{C}_3 = \rho\bar{u}F$$

The relation between π_{xx} and $p = \rho RT$ may be written as

$$\rho\,\frac{\gamma - 1}{2\gamma}(1 - \bar{u}^2) = \frac{\pi_{xx}}{{u_{max}}^2} + 2C_1\sqrt{1 - \bar{u}^2}\,\frac{d\bar{u}}{dx}$$

$$\frac{d\bar{u}}{dx} = \frac{1}{\sqrt{1 - \bar{u}^2}}\left[\frac{\gamma - 1}{4\gamma C_1}\,\frac{\bar{C}_3(1 - \bar{u}^2)}{\bar{u}F} - \frac{\pi_{xx}}{{u_{max}}^2}\,\frac{1}{2C_1}\right]$$

If the preceding expression for $\frac{d\bar{u}}{dx}$ is substituted into the relation for $\frac{d^2\log F}{dx^2}$, the modified form of the Navier-Stokes equation is obtained:

$$\frac{d^2\log F}{dx^2} = \left(\frac{\bar{C}_3}{2C_1\sqrt{1 - \bar{u}^2}} - \frac{dF}{dx}\right)\frac{1}{\bar{u}F\sqrt{1 - \bar{u}^2}}\left(\frac{\gamma - 1}{4\gamma C_1}\,\bar{C}_3\,\frac{1 - \bar{u}^2}{\bar{u}F} - \frac{\pi_{xx}}{{u_{max}}^2}\,\frac{1}{2C_1}\right) +$$

$$\frac{1}{2C_1\sqrt{1 - \bar{u}^2}\,\bar{u}}\,\frac{1}{{u_{max}}^2}\,\frac{d\pi_{xx}}{dx} = 0 \qquad (19)$$

For determining the connection of a curved minimum passage to the sink flow (given by curve 4), it should be recalled that the reason that the sink flow could be continued only up to a certain distance beyond the isentropic minimum section was the breakdown of the Navier-Stokes equation. The shape of a nonisentropic minimum passage is thus limited by the requirement that $\pi_{xx} > 0$. Since, beyond the isentropic minimum section, the Navier-Stokes equation is close to the breakdown criterion (see table I, curve 4), the condition for the minimum passage will be introduced in equation (19) that $\pi_{xx} = \text{Constant}$ or $\dfrac{d\pi_{xx}}{dx} = 0$.

In order to determine in principle if the flow will pass through a smaller than isentropic curved minimum section, it appeared more expedient to investigate the fundamental behavior of equation (19) by simple inspection rather than by integration. Since the entropy balance (appendix C) is indirectly contained in the modified Navier-Stokes equation, $\dfrac{d\overline{u}}{dx}$ has to be positive; furthermore, for the converging part of the minimum passage $\dfrac{dF}{dx}$ is negative. Thus, if $\overline{u}$ does not exceed its physical limits and thus invalidates the result (this problem is discussed subsequently), $\dfrac{d^2 \log F}{dx^2}$ will remain positive. In order to understand the meaning of this condition, the variation of the cross section F corresponding to the simple cases $\dfrac{d^2 \log F}{dx^2} = \text{Constant}$ and $\dfrac{d^2 \log F}{dx^2} = 0$ will be briefly discussed. Integration of $\dfrac{d^2 \log F}{dx^2} = \text{Constant} = A$ results in

$$F = Ce^{\frac{Ax^2}{2}+Bx}$$

where A, B, and C are constants.

Now, since at the junction of the converging wedge and the connecting piece $\dfrac{d \log F}{dx}$ is negative, B will be negative. The quantity A is positive since $\dfrac{d^2 \log F}{dx^2}$ has to be positive. The quantity e raised to the negative value of a power of x will be represented by a curve asymptotic to the positive x-axis. However, the quantity e raised to the positive value of the power of x will be asymptotic to the negative x-axis and, thus, will positively increase. Since the positive contribution is connected with x^2 and the negative contribution with x, the positive contribution will be the deciding one and $\frac{d^2 \log F}{dx^2}$ will form a variation of cross section which has a minimum section and increases downstream of it. The condition $\frac{d^2 \log F}{dx^2} = 0$ which does not agree with the requirement of a positive $\frac{d\bar{u}}{dx}$ will yield

$$F = Ce^{Bx}$$

where a negative B represents a curve asymptotic to the positive x-axis.

Finally, it is investigated by inspection whether $\bar{u}$ exceeds its physical limits. A lack of a limiting value of $\bar{u}$, coupled with the continuity condition and other available relations, will indicate that the other flow variables will not reach limiting values. Thus, the problem arises whether the requirement of a positive $\frac{d\bar{u}}{dx}$ may cause $\bar{u}$ to increase to its limiting value $\bar{u} = 1$. If $\bar{u}$ approaches 1, the first term in the expression for $\frac{d\bar{u}}{dx}$, upon eliminating the brackets, will go to 0 and the second term will go to infinity; since the second term is subtracted from the first term, this behavior means that as $\bar{u}$ goes to 1, $\frac{d\bar{u}}{dx}$ becomes negative, which again indicates that actually $\bar{u} = 1$ will not be reached. A check of whether $\frac{d\bar{u}}{dx}$ will actually remain positive as indicated by the entropy balance (indirectly contained in equation (19)) can be made to some extent by simple inspection of equation (19). (Such a check will also show if any unexpected irregularities occur in equation (19).) The term in parentheses

$$\left(\frac{\gamma - 1}{4\gamma C_1} \bar{C}_3 \frac{1 - \bar{u}^2}{\bar{u}F} - \frac{\text{Constant}}{2C_1}\right)$$

indicates that for $\bar{u} = \text{Constant}$ and decreasing F the first term in the parentheses will increase, and since the second term is a constant, $\frac{d\bar{u}}{dx}$ will remain positive. The fact that the first term includes $1 - \bar{u}^2$ does not affect the considerations since $\bar{u} = \text{Constant}$. These considerations do not yield information concerning the minimum section itself for which $\frac{dF}{dx} = 0$ or for increasing cross sections. For these conditions, as for all other states at cross sections smaller than the isentropic minimum section, the entropy balance, as previously stated, yields sufficient information for the present purpose. For increasing cross section, the first parenthetic term on the right side of equation (19) will tend to reach 0 after a while. A more exact investigation of the case of variable cross section is not within the scope of the present paper.

Similar to the flow passage through a minimum section of sink flow alone, for the case of flow through a curved minimum section joined to a sink flow, the flow passage through a cross section smaller than the isentropic minimum cross section is possible. In other words, a larger mass flow may pass through a given isentropic minimum cross section than for isentropic flow. The significance of the result can be amplified by a comparison of the conditions of mass flow through a minimum section for the case of free-molecular flow and the present viscous-flow case for which the transverse viscous effects are neglected. Since the results obtained in the paper apply not only to minimum sections of source or sink flow but to curved minimum sections in general, an especially simple case of free-molecular flow can be chosen for a comparison. It is the case of effusive molecular flow from a vessel (reference 6), for which, similar to the present viscous flow, no shear at the flow boundaries will exist. The reason that free-molecular source or sink flow could not be chosen for comparison is that the conditions of molecular chaos on which the Maxwell distribution is based are no longer fulfilled for source or sink flow where all molecules have to leave or enter a given point (three dimensional) or line (two dimensional). The mass flow for the special case of effusive free-molecular flow is (reference 6)

$$\frac{1}{4} c_o \rho_o = \frac{1}{4}\sqrt{\frac{8}{\pi\gamma}}\, a_o \rho_o = 0.337 a_o \rho_o \qquad (\text{for air, } \gamma = 1.4)$$

where the subscript o refers to stagnation conditions. The isentropic mass flow through a minimum section is given by (see reference 7)

$$0.579 a_o \rho_o \qquad (\text{for air, } \gamma = 1.4)$$

Thus, the "very viscous" effusive free-molecular flow has a smaller mass flow than the isentropic flow. In contrast, the present solution of the Navier-Stokes equation, dealing only with the longitudinal viscous effects, may have a higher mass flow. Thus in this respect, the viscous flow does not follow the trend of the free-molecular flow. The main reason for this exceptional behavior of the flow apparently lies in its ability to send energy ahead (by heat conduction), which cannot occur for isentropic flow. The quantitative nature of this trend still has to be determined, namely, whether the trend towards larger mass flow indicated by the present solution of the Navier-Stokes equation will yield much higher, or negligibly higher, maximum values than the isentropic flow before the mass flow drops to the value for free-molecular flow. In order to check this trend quantitatively, a more exact relation than the Navier-Stokes equation should be investigated.

CONCLUSIONS

A study of the solution of the Navier-Stokes equations for source and sink flows of a viscous, heat-conducting, compressible fluid yields the following conclusions:

1. The fundamental effect of viscosity and heat conduction is a smoothing out of the flow discontinuities which exist when the viscous effects due to source flow are zero. Such smoothing effects of flow discontinuities are well-known in boundary-layer theory and one-dimensional shock flow theory.

2. The influence of the familiar heat-conduction effects combined with the longitudinal viscous effects alone (no transverse viscous effects) on a flow with constant total energy may cause a larger mass flow to pass through a given isentropic minimum cross section of a sink flow than for isentropic expansion flow. The reason for such a surprising behavior is that heat may be conducted downstream. The same effect applies also to flow through a curved minimum section joined to a sink flow. This trend does not follow the comparable case of free-molecular flow through a minimum section, since the free-molecular flow will have a smaller mass flow than the corresponding isentropic case. The quantitative nature of this different trend should be investigated with relations more exact than the Navier-Stokes equation.

3. The solutions for viscous, heat-conducting, compressible fluid require larger area ratios for the passage of a fluid through a given velocity change than do the corresponding solutions for the cases that the viscous effects due to source or sink flow are zero.

4. The order of magnitude of the longitudinal viscous effects is indicated by a dimensionless parameter having the form of a reciprocal Reynolds number. This parameter is independent of position from the source or sink for two-dimensional flow and increases with the distance

for three-dimensional flow. The parameter indicates that the isolated longitudinal viscous effects should be negligible for hypersonic tunnels operating in the range of atmospheric stagnation conditions even though the minimum section of the tunnel may be small based on engineering standards. If the minimum section of the tunnel is to influence greatly the isolated longitudinal viscous effects without becoming impracticably small, the mean free path has to be increased to reach the order of the width of the minimum section; that is, the stagnation density of the flow has to be correspondingly low.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., October 26, 1951

APPENDIX A

DISCUSSION OF THE SECOND VISCOSITY COEFFICIENT μ' FROM THE GAS-DYNAMIC APPROACH

The expression for the second viscosity coefficient μ', given by Busemann in reference 2, is

$$\mu' = \frac{n - 3}{n} \mu \tag{A1}$$

where n represents the number of degrees of freedom and is obtained by comparing the expressions for the normal viscous stress τ_{xx} based on continuum theory and kinetic-gas theory. The two expressions are

$$\tau_{xx} = -2\mu \frac{\partial u}{\partial x} - \frac{2}{3}(\mu' - \mu)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{A2}$$

and

$$\tau_{xx} = -2K_2 \rho c l\left[\frac{\partial u}{\partial x} - \frac{1}{n}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] \tag{A3}$$

where K_2 is a constant.

These equations indicate that if only the three translational degrees of freedom of a molecule have to be considered, that is, $n = 3$, μ' will be 0. Since the existence of μ' is due to excitation of the internal degrees of freedom, a brief discussion of its relation to the well-established time-lag effect should be of interest. The use of this effect in gas dynamics is discussed in references 15 and 16, and the application to ultrasonics is given in reference 17. (Also see bibliography of reference 17.) The expression of μ' through the time-lag effect is obtained by modifying the effect in such a manner that it is in agreement with the fundamental assumptions inherent in the Navier-Stokes equation. Tisza shows (reference 3) that for the case of ultrasonics the time lag may only be incorporated in the Navier-Stokes equation if $\omega t \ll 1$, where t is the time lag of the internal degrees of freedom and ω is the sound frequency. For gas dynamics, this relation is conveniently written in the form

$$\frac{t_{int}}{t_{flow}} \ll 1$$

that is, as the ratio of the time lag of the internal degrees of freedom and the time in which the change in flow variables, for example, the velocity, occurs. In other words, this relation indicates that t_{int} may be large, provided t_{flow} is much larger. The limit of applicability of μ' (analagous to the previously discussed limit of applicability of the Navier-Stokes equation) is assumed to be reached when t_{int} is of the same order as t_{flow}.

The detailed derivation of equation (A3) based on kinetic-gas theory is presented as follows: Equation (A3) represents the normal stress in the x-direction acting on a three-dimensional compressible fluid element due to a small deviation from the equilibrium state or the Maxwell distribution, for the case that the deviation causes no transverse viscous effects. It is convenient to show first the effects of a deviation for a gas with one degree of freedom. The equilibrium state is given by the pressure (reference 18 which is helpful for an understanding of the entire subsequent derivation),

$$p_x = \rho_x c_x^2$$

The subscript x indicates the x-direction. Note that ρ_x is a line density. The deviation from the equilibrium state is

$$dp_x = 2\rho_x c_x \, dc_x + c_x^2 d\rho_x$$

For the definition of c_x, in view of the general nature of the problem no distinction is made between various methods of averaging. The expression for the deviation dp_x is simply made to apply to a three-dimensional gas by using the volume density ρ instead of just the line density ρ_x. The average velocity c_x is not affected by this generalization to three dimensions, since the three-dimensional Maxwell distribution may be obtained by superposing three independent one-dimensional equilibrium distributions in the x-, y-, and z-directions. This independence is also expressed by

$$c_x^2 = \frac{c^2}{3}$$

The deviation dp_x, which makes use of the correct volume density, may be written

$$\left(dp_x\right)_{vol} = \frac{1}{3}\left(2\rho c^2 \frac{dc_x}{c_x} + c^2 d\rho\right)$$

The term $d\rho$ may be expressed by the continuity equation in the form

$$\frac{d\rho}{dt} = -\rho \operatorname{div} \overline{v}$$

In order to make the results obtained by kinetic theory comparable to those of continuum theory, $\frac{dc_x}{c_x}$ is expressed in terms of the velocity gradient $\frac{\partial u}{\partial x}$. For the case of small deviations, the following relation is true:

$$\frac{dc_x}{c_x} = -\frac{du}{u}$$

or since $u = \frac{dx}{dt}$

$$\frac{dc_x}{c_x} = -\frac{du}{dx} dt$$

Substituting these relations in $\left(dp_x\right)_{vol}$ gives

$$\left(dp_x\right)_{vol} = \frac{1}{3}\left(-2\rho c^2 \frac{\partial u}{\partial x} - c^2 \rho \operatorname{div} \overline{v}\right)dt$$

Since the stress τ_{xx} represents only the irreversible effect of the deviation from the equilibrium state, the effect of the reversible or isentropic deviation has to be subtracted from the preceding expression. In contrast for stress based on purely transverse viscous effects, a deviation from equilibrium can only cause irreversible effects. The reversible effect of the deviation is given by

$$\left(\frac{dp}{p}\right)_{rev} = \gamma \frac{d\rho}{\rho}$$

With the use of the continuity condition and

$$\gamma = \frac{2 + n}{n}$$

the following equation can be written

$$(dp)_{rev} = - \frac{2 + n}{n} p \operatorname{div} \overline{v} \, dt = - \frac{2 + n}{n} \frac{1}{3} \rho c^2 \operatorname{div} \overline{v} \, dt$$

Subtracting $(dp)_{rev}$ from $(dp_x)_{vol}$ as previously suggested gives

$$\begin{aligned}(dp_x)_{vol} - (dp)_{rev} &= - \frac{1}{3} \rho c^2 \left[2 \frac{\partial u}{\partial x} + \left(1 - \frac{2 + n}{n}\right) \operatorname{div} \overline{v}\right] dt \\ &= -2\left(\frac{1}{3} \rho c^2\right)\left(\frac{\partial u}{\partial x} - \frac{1}{n} \operatorname{div} \overline{v}\right) dt\end{aligned}$$

In order to obtain the stress τ_{xx}, as given by equation (A3), integrate the preceding equation with respect to the time t. For the present purpose, the integration can be performed to a good approximation by changing dt to Δt which can be done for the following reasons: The integration with respect to time indicates that the magnitude of the deviation τ_{xx} from the reversible reference level will depend on the time the deviation can be regarded as independent of the effects of collisions. The simplification of the integration by the use of Δt means that a certain average value is assumed for the deviation which is abruptly terminated after a time Δt when the energy due to the velocity change in one direction $\left(\text{see term } \frac{\partial u}{\partial x}\right)$ is suddenly distributed over all degrees of freedom $\left(\text{see term } \frac{1}{n} \operatorname{div} \overline{v}\right)$. The time Δt, in accordance with basic concepts of kinetic theory, is approximately given by the ratio l/c where l is the mean free path. The introduction of a finite time does not violate the assumption of small deviations in the Navier-Stokes equation. As previously shown, the immediate neighborhood is a relative concept, based on the requirement that the ratio $\frac{t_{int}}{t_{flow}} \ll 1$. For this case the time lag of the internal degrees of freedom t_{int} equals that of the translational degrees. The stress τ_{xx} may be written thus

$$\tau_{xx} = -2\left(\frac{1}{3}\rho c l\right)\left(\frac{\partial u}{\partial x} - \frac{1}{n}\operatorname{div}\overline{v}\right) \tag{A4}$$

The expression is identical with equation (A3) if K_2 is set equal to 1/3.

In accordance with a suggestion given in reference 2, the preceding equation (A4) is generalized by taking into account the fact that not all the molecular degrees of freedom will be excited after a single collision or after a single mean free path:

$$\tau_{xx} = -2\mu\left[\frac{l_1}{l_1}\left(\frac{1}{1}\frac{\partial u}{\partial x} - \frac{1}{3}\operatorname{div}\overline{v}\right) + \frac{l_2}{l_1}\left(\frac{1}{n_1} - \frac{1}{n_2}\right)\operatorname{div}\overline{v} + \frac{l_3}{l_1}\left(\frac{1}{n_2} - \frac{1}{n_3}\right)\operatorname{div}\overline{v} + \ldots\right] \tag{A5}$$

where the subscripts 2 and 3 refer to other than translational degrees of freedom and μ is given by $K_2\rho c l$. Now, an expression for μ' can be obtained by comparing the expression (A5) with equation (A2), rewritten in the following convenient form:

$$\tau_{xx} = -2\mu\left[\frac{\partial u}{\partial x} + \frac{1}{3}\left(\frac{\mu'}{\mu} - 1\right)\operatorname{div}\overline{v}\right]$$

Comparison of the preceding two expression for τ_{xx} yields

$$\frac{1}{3}\left(\frac{\mu'}{\mu} - 1\right) = -\frac{1}{3} + \frac{l_2}{l_1}\left(\frac{1}{n_1} - \frac{1}{n_2}\right) + \frac{l_3}{l_1}\left(\frac{1}{n_2} - \frac{1}{n_3}\right) + \ldots$$

For μ'

$$\frac{\mu'}{\mu} = 3\left[\frac{l_2}{l_1}\left(\frac{1}{n_1} - \frac{1}{n_2}\right) + \frac{l_3}{l_1}\left(\frac{1}{n_2} - \frac{1}{n_3}\right) + \ldots\right]$$

or

$$\frac{\mu'}{\mu} = 3\left[\frac{l_2}{l_1}\left(\frac{n_2 - n_1}{n_1 n_2}\right) + \frac{l_3}{l_1}\left(\frac{n_3 - n_2}{n_2 n_3}\right) + \ldots\right]$$

For purposes of demonstration it is of interest to make a rough estimate of the numerical value of the expression for μ'/μ for a diatomic gas at room temperature. The three translational degrees $(n_1 = 3)$ and the two rotational degrees (total $n_2 = 5$) are excited after one collision $\left(\frac{l_2}{l_1} = 1\right)$. For room temperature, the number of collisions or the number of mean free paths necessary to excite the vibrational energy is very large, for example, of the order of 10^5 (for a gas without impurities). The vibrational energy at room temperature is, however, only a small fraction of a percent of the vibrational energy at equipartition; thus, in terms of the expression for μ'/μ not two vibrational degrees (total $n_3 = 7$) are excited but only a very small fraction (total n_3 = Very small fraction over 5). The contribution of the vibrational term, the product of a large and a small quantity, can thus be expected to be of the order of magnitude of one. Note further that because of the limit of $\frac{t_{int}}{t_{flow}} \ll 1$ to the validity of μ' for very rapid flow changes like shocks (t_{flow} is small), it may be impossible to incorporate the effect of vibrational energy excited after many collisions (t_{int} is large) in μ'. Now, for example, assume that the rotational degrees are fully excited after two collisions or mean free paths and that the contributions of the vibrational energy to μ' is zero

$$\frac{\mu'}{\mu} = (3)(2)\frac{2}{15} = \frac{4}{5}$$

The estimates give meaning to the use made in the present paper of a μ'/μ of the order of unity for air undergoing slow and rapid flow changes.

APPENDIX B

SOURCE FLOW WITH HEAT ADDITION

For the general case of source flow with heat addition or variable total energy, the energy equation can no longer be used separately as was possible for source flow with constant total energy; rather, the equations of continuity, of motion, and of energy have to be integrated simultaneously. Furthermore, the amount of total energy addition or heat addition to the flow is no longer treated as a separable boundary condition, but all boundary conditions necessary to satisfy the system of differential equations have to be satisfied simultaneously. An exception is known to exist for the case of one-dimensional shock flow with heat addition (references 11, 12, and 19) where a separate integration of the energy equation is made possible by use of certain values of the Prandtl number depending on the ratio μ'/μ. This fact can also be seen from equation (7) in the present paper when F is taken constant.

In the body of this paper the exponential deviations for terminated source flow from the main constant-total-energy solutions (curves 1 and 4) were discussed in some detail. Similarly, terminating deviations exist because of a heat source; however, as just indicated, for source flow the heat addition to the flow may no longer be treated as a separable boundary condition. In view of this complication, a much simpler problem is chosen to illustrate merely the basic nature of the effects caused by flow termination with a heat source. Assume that a wire is pulled with the speed u through a heat source at a fixed location. The differential equation for heat conduction "upstream" in the wire (the "downstream" parts are heated as the wire is drawn through the source) is given by

$$-u \frac{dT}{dx} = \frac{K}{\rho c_p} \frac{d^2T}{dx^2}$$

Integration indicates that a dying out of the heat-source effects occurs:

$$T - T_\infty = \text{Constant}\left(\frac{K}{\rho u c_p} e^{-\frac{\rho u c_p}{K} x}\right)$$

where T_∞ is the temperature of the unheated wire.

In order to complete the physical picture, an estimate is made of the distance which the major effects, caused by the terminating heat source, will penetrate "upstream"; the total effects will, of course, be conducted to ∞. In accordance with customary procedure (used, for example, for Prandtl's estimate of the shock thickness), the subtangent of the actual variation is taken as a measure of the penetration of the major effects. For the exponential temperature variation given in the preceding equation, the subtangent is

$$\frac{K}{\rho u c_p}$$

A rough estimate of this distance can be made by assuming that this measure holds approximately true for gas flow. If the various constants are left out from the pertinent expressions of the kinetic-gas theory (reference 6), the distance of penetration can be written as

$$\frac{K}{\rho u c_p} \approx \frac{\rho a c_p}{\rho u c_p} = \frac{l}{M}$$

Attention is drawn to the fact that for this rough estimate in the neighborhood of Mach number of 1, a Mach number passed through by every shock, the penetration distance of the major effects of the terminating heat source is of the order of the mean free path.

APPENDIX C

ENTROPY BALANCE

General Development

The discussion of the entropy balance is conveniently introduced by the complete energy balance for the same fluid element. The heat dQ added to the fluid element may be separated into the heat added from the outside by conduction dQ_{cond} and the heat input due to internal generation of heat by friction. The energy balance is expressed in terms of the familiar dissipation function ϕ in the form (references 2, 20, and 21)

$$\rho \frac{dQ}{dt} - \rho \frac{dQ_{cond}}{dt} = \phi \tag{C1}$$

The term $\frac{dQ_{cond}}{dt}$ is known from the law of heat conduction. The inflow or outflow of heat on one side of the element is

$$-k \text{ grad } T$$

and the rate of heat addition by longitudinal heat conduction through two sides of the element (for source flow heat is not conducted transversally to the flow) is

$$\rho \frac{dQ_{cond}}{dt} = \text{div}(k \text{ grad } T)$$

For the purpose of subsequent comparison with the entropy balance, the energy balance is divided by T

$$\rho \frac{ds}{dt} - \frac{1}{T} \text{div}(k \text{ grad } T) = \frac{\phi}{T} \tag{C2}$$

The entropy balance is made for an isolated system (references 22 and 23). The entropy s of the fluid element is that of an open system since heat exchange by conduction occurs with the neighboring elements. The system can be made an isolated one by arranging at the boundaries of the fluid element an artificial entropy storage which compensates for the outflow of heat. The entropy flow through one wall of the fluid element is given by

$$\frac{-k \text{ grad } T}{T}$$

in contrast to the corresponding heat flow by conduction $-k$ grad T. The entropy stores furnish for the entire fluid element the entropy increase rate

$$\text{div}\left(\frac{k \text{ grad } T}{T}\right)$$

Since, now the system is isolated, the second law may be expressed by the inequality

$$\rho \frac{ds}{dt} - \text{div}\left(\frac{k \text{ grad } T}{T}\right) \geqq 0 \qquad \text{(C3)}$$

To obtain the entropy balance in a more convenient form, the second term is developed with the aid of the general vector relation

$$\text{div}(a\bar{b}) = \text{grad } a.\bar{b} + a \text{ div } \bar{b}$$

where the vector $\bar{b} = k \text{ grad } T$ and the scalar $a = \frac{1}{T}$. Therefore, the entropy balance becomes

$$\rho \frac{ds}{dt} - \frac{\text{div}(k \text{ grad } T)}{T} + \frac{k}{T^2}(\text{grad } T)^2 \geqq 0 \qquad \text{(C4)}$$

or with the use of equation (C2)

$$\rho \frac{ds}{dt} - \text{div}\left(\frac{k \text{ grad } T}{T}\right) = \frac{\phi}{T} + \frac{k}{T^2}(\text{grad } T)^2 \geqq 0 \qquad \text{(C5)}$$

Since $\phi \geqq 0$ (references 2, 20, and 21) and naturally k and T are positive, the correctness of the inequality is evident.

For the case that the total velocity of a steady flow is in the symmetry axis of the flow, the term $\rho \frac{ds}{dt}$ becomes equal to $\rho u \frac{ds}{dx}$. Another form of the entropy balance may be obtained by expressing $\rho u \frac{ds}{dx}$ in equation (C5) as a divergence which can be done by showing that

$$\rho u \frac{ds}{dx} = \mathrm{div}(\rho\overline{v}s)$$

Specifically,

$$\mathrm{div}(\rho\overline{v}s) = \rho\overline{v} \cdot \mathrm{grad}\ s + s\ \mathrm{div}(\rho\overline{v})$$

The steady-flow continuity condition, however, expresses

$$\mathrm{div}(\rho\overline{v}) = 0$$

and, thus,

$$\mathrm{div}(\rho\overline{v}s) = \rho\overline{v} \cdot \mathrm{grad}\ s = \rho u \frac{ds}{dx}$$

Equation (C5) may be rewritten as

$$\mathrm{div}\left(\rho\overline{v}s - \frac{k\ \mathrm{grad}\ T}{T}\right) \geqq 0$$

For source flow or for flow through slightly varying cross section,

$$\frac{1}{F}\frac{d}{dx} F\left(\rho us - \frac{k}{T}\frac{dT}{dx}\right) \geqq 0$$

or, since $\rho uF = \mathrm{Constant}$,

$$s_2 - \left(\frac{k}{\rho uT}\frac{dT}{dx}\right)_2 \geqq s_1 - \left(\frac{k}{\rho uT}\frac{dT}{dx}\right)_1 \geqq s_0 \qquad \text{(C6)}$$

where 0, 1, 2 represent subsequent points in the downstream direction.

For the case of shock flow with constant total energy in an infinite tube with constant cross section, it is not necessary to resort to this more complicated entropy balance. The reason is that, if in front ($-\infty$) and behind ($+\infty$) the shock uniform conditions exist (that is, the velocity gradient and, thus, the temperature gradient are zero), no heat conduction will occur there, and thus, the shock as a whole may be regarded as an isolated system. This analysis is in essential agreement with a brief general statement in reference 19 concerning the reason why a negative entropy gradient occurring inside the shock (where a fluid element is an open system) does not violate the second law. The function of heat conduction in the entropy balance is, however, not directly mentioned in reference 19.

Flow through Cross Sections Smaller than the Isentropic Minimum Section

It is well known that a certain isoenergetic mass flow with a given entropy can pass only through a certain minimum cross section. The physical reason for the penetration of an isoenergetic solution to cross sections smaller than the isentropic minimum of a sink flow, due to expansion in the continued sink flow, or in a curved minimum passage joined to the sink flow may be disclosed through the entropy balance, which is indirectly contained in the Navier-Stokes equation. The entropy balance is expressed by the inequality (C6), where the subscript 0 would correspond to the isentropic minimum section and subscripts 1 and 2 would refer to cross sections farther downstream. Since the problem is one of constant total energy, the entropies in the smaller downstream cross sections have to be progressively smaller than in the isentropic minimum section (references 2 and 24). Thus, in order that the inequality (C6) be satisfied, $\left(\frac{K}{\rho u T}\frac{dT}{dx}\right)_2$ has to be greater than $\left(\frac{K}{\rho u T}\frac{dT}{dx}\right)_1$ and, furthermore, both expressions have to be negative. Since for the present case of constant-energy flow $\frac{dT}{dx}$ is the negative of $\frac{du}{dx}$ multiplied by a function of u, $\frac{du}{dx}$ has to be positive, and, in addition, it has to be sufficiently large as a consequence of the second law of thermodynamics. Since the entropy balance is indirectly contained in the present solutions of the Navier-Stokes equation, the expansion curve 4 which penetrates beyond the isentropic minimum section has to meet these requirements of the entropy balance, provided, of course, that the Navier-Stokes equation is not used beyond the limit of its applicability. The entropy balance also indicates that a supersonic compression flow due to its positive temperature gradient will not be able to pass through a smaller than isentropic minimum section.

APPENDIX D

REDUCTION OF SOURCE-FLOW EQUATIONS TO SHOCK FLOW IN CONSTANT CROSS SECTION

The case of a shock in constant cross section will be investigated with the aid of equation (14) which is rewritten

$$\xi'' = -\frac{\bar{u}}{1 - \bar{u}^2}\,\xi' - \frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{\frac{4\gamma C_2}{\gamma - 1}\bar{u}^2\sqrt{1 - \bar{u}^2}}\,\xi'^2 + \left(\frac{\gamma - 1}{4\gamma C_2}\,\frac{\sqrt{1 - \bar{u}^2}}{\bar{u}} - \bar{u}\right)\xi'^3 \tag{D1}$$

where

$$\xi' = \frac{d \log x}{d\bar{u}} = \frac{1}{x}\,\frac{dx}{d\bar{u}}$$

Since for flow through constant cross section x is infinite, it is eliminated from equation (D1) by dividing by $C_2 = \frac{C_1}{\rho\bar{u}x}$ which gives

$$\frac{\xi''}{C_2} = -\frac{\bar{u}}{1 - \bar{u}^2}\,\frac{\xi'}{C_2} - \frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{\frac{4\gamma}{\gamma - 1}\bar{u}^2\sqrt{1 - \bar{u}^2}}\,\frac{\xi'^2}{C_2{}^2} + \left(\frac{\gamma - 1}{4\gamma}\,\frac{\sqrt{1 - \bar{u}^2}}{\bar{u}} - C_2\bar{u}\right)\frac{\xi'^3}{C_2{}^2} \tag{D2}$$

For the present flow $x = \infty$ and $\rho\bar{u} = \text{Constant}$; thus

$$\frac{\xi'^3}{C_2{}^2} = \frac{1}{x^3}\left(\frac{dx}{d\bar{u}}\right)^3 \frac{1}{\left(\frac{C_1}{\rho\bar{u}x}\right)^2} = 0$$

Introducing the new variable

$$\eta = \frac{\xi'}{C_2} = \frac{\rho\bar{u}}{C_1}\,\frac{dx}{d\bar{u}} = C_4\,\frac{dx}{d\bar{u}}$$

gives

$$\eta' = -\frac{\bar{u}}{1 - \bar{u}^2}\eta - \frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{\frac{4\gamma}{\gamma - 1}\bar{u}^2\sqrt{1 - \bar{u}^2}}\eta^2 \tag{D3}$$

The integration of equation (D3) is accomplished by transforming it in the following manner: Since

$$d\sqrt{1 - \bar{u}^2} = -\frac{\bar{u}\ d\bar{u}}{\sqrt{1 - \bar{u}^2}}$$

$d\bar{u}$ can be expressed as

$$d\bar{u} = -\frac{\sqrt{1 - \bar{u}^2}}{\bar{u}}\ d\sqrt{1 - \bar{u}^2}$$

Now, equation (D3) can be written

$$d\eta = \eta\left(-\frac{\bar{u}}{1 - \bar{u}^2}\right)\left(\frac{-\sqrt{1 - \bar{u}^2}}{\bar{u}}\right)d\sqrt{1 - \bar{u}^2} - \eta^2\frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{\frac{4\gamma}{\gamma - 1}\bar{u}^2\sqrt{1 - \bar{u}^2}}\left(-\frac{\sqrt{1 - \bar{u}^2}}{\bar{u}}\right)d\sqrt{1 - \bar{u}^2}$$

or

$$d\eta = \frac{\eta}{\sqrt{1 - \bar{u}^2}}\ d\sqrt{1 - \bar{u}^2} + \eta^2\frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{\frac{4\gamma}{\gamma - 1}\bar{u}^3}\ d\sqrt{1 - \bar{u}^2}$$

Multiplying the preceding equation by $\sqrt{1 - \bar{u}^2}\Big/\eta^2$ gives

$$\frac{\sqrt{1 - \bar{u}^2}\ d\eta - d\sqrt{1 - \bar{u}^2}\ \eta}{\eta^2} = \frac{\frac{\gamma + 1}{\gamma - 1}\bar{u}^2 - 1}{\frac{4\gamma}{\gamma - 1}\bar{u}^3}\sqrt{1 - \bar{u}^2}\ d\sqrt{1 - \bar{u}^2}$$

or, since $d\left(\sqrt{1 - \bar{u}^2}\right) = - \frac{\bar{u}\, d\bar{u}}{\sqrt{1 - \bar{u}^2}}$,

$$-d\left(\frac{\sqrt{1 - \bar{u}^2}}{\eta}\right) = - \frac{\frac{\gamma + 1}{\gamma - 1} \bar{u}^2 - 1}{\frac{4\gamma}{\gamma - 1} \bar{u}^2} d\bar{u}$$

From the integration of the preceding equation, the following equation results

$$\frac{\sqrt{1 - \bar{u}^2}}{\eta} = \int \frac{\frac{\gamma + 1}{\gamma - 1} \bar{u}^2 - 1}{\frac{4\gamma}{\gamma - 1} \bar{u}^2} d\bar{u} + \text{Constant}$$

$$= \int \left(\frac{\gamma + 1}{4\gamma} - \frac{\gamma - 1}{4\gamma} \frac{1}{\bar{u}^2}\right) d\bar{u} + \text{Constant}$$

or

$$\frac{\sqrt{1 - \bar{u}^2}}{\eta} = \frac{\gamma + 1}{4\gamma} \bar{u} + \frac{\gamma - 1}{4\gamma} \frac{1}{\bar{u}} + \text{Constant}$$

or

$$\eta = \frac{\sqrt{1 - \bar{u}^2}}{\frac{\gamma + 1}{4\gamma} \bar{u} + \frac{\gamma - 1}{4\gamma} \frac{1}{\bar{u}} + \text{Constant}}$$

Since in the final expression for constant-energy flow through the shock in constant cross section the velocity gradient $\frac{1}{\eta} = \frac{1}{C_4}\frac{d\bar{u}}{dx}$ has to become zero, two infinities are to be expected for η as the uniform velocities $\bar{u}_1$ and $\bar{u}_2$ in front of and behind the shock, respectively, occur. This condition requires an expression of the type $(\bar{u} - \bar{u}_1)(\bar{u} - \bar{u}_2)$ or a quadratic form in the denominator. The quadratic form is obtained by multiplying and dividing the preceding equation by $\bar{u}$ which gives

$$\eta = \frac{\sqrt{1 - \bar{u}^2}\,\bar{u}}{\frac{\gamma + 1}{4\gamma}\bar{u}^2 + \frac{\gamma - 1}{4\gamma} + \text{Constant }(\bar{u})}$$

The denominator may also be written in the form

$$\frac{\gamma + 1}{4\gamma}\left[\bar{u}^2 + \frac{\gamma - 1}{\gamma + 1} + \text{Constant }(\bar{u})\right]$$

Furthermore, across the shock, according to Prandtl, the relation

$$u_1 u_2 = a*^2$$

exists where $a*$ is the critical velocity. Since $\bar{u}_1$ and $\bar{u}_2$ represent the ratios u_1/u_{max} and u_2/u_{max}

$$\bar{u}_1\bar{u}_2 = \frac{\gamma - 1}{\gamma + 1}$$

and the denominator may be written as

$$\frac{\gamma + 1}{4\gamma}\left[\bar{u}^2 + \bar{u}_1\bar{u}_2 + \text{Constant }(\bar{u})\right]$$

Since

$$\left(\bar{u} - \bar{u}_1\right)\left(\bar{u} - \bar{u}_2\right) = \bar{u}^2 - \bar{u}\left(\bar{u}_1 + \bar{u}_2\right) + \bar{u}_1\bar{u}_2$$

the expression inside the brackets represents $\left(\bar{u} - \bar{u}_1\right)\left(\bar{u} - \bar{u}_2\right)$ and thus satisfies the conditions ahead of and behind the shock, if Constant $= -\left(\bar{u}_1 + \bar{u}_2\right)$. The following expression, thus, may be written for x since $\eta = C_4 \frac{dx}{d\bar{u}}$:

$$x = \frac{1}{C_4} \frac{4\gamma}{\gamma + 1} \int \frac{\sqrt{1 - \bar{u}^2}\, \bar{u}\, d\bar{u}}{\left(\bar{u} - \bar{u}_1\right)\left(\bar{u} - \bar{u}_2\right)} + \text{Constant}$$

This expression essentially agrees with the known expression for the velocity variation across a one-dimensional shock. (In comparing the result with that of reference 19, note that $\mu' = 0$ in the reference and that the variables used are not the same as those of the present paper.)

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TABLE I

RATIOS OF $\left(\dfrac{p}{2\mu \dfrac{du}{dx}}\right)_{Plot}$ GIVEN BY PLOTTED SOLUTION OF NAVIER-STOKES EQUATION TO BREAKDOWN

VALUES OF $\left(\dfrac{p}{2\mu \dfrac{du}{dx}}\right)_{Breakdown}$ OF NAVIER-STOKES EQUATION

[Plotted solutions are given in figures 1 to 3.]

$\overline{u}$	$\phi' \equiv \dfrac{d \log x}{d \log \overline{u}}$	$\dfrac{1}{\phi'} \equiv \left(\dfrac{d \log \overline{u}}{d \log x}\right)_{Plot}$	$\left(\dfrac{d \log \overline{u}}{d \log x}\right)_{Breakdown}$	$\dfrac{\left(\dfrac{p}{2\mu \frac{du}{dx}}\right)_{Plot}}{\left(\dfrac{p}{2\mu \frac{du}{dx}}\right)_{Breakdown}} \equiv \dfrac{\left(\dfrac{d \log \overline{u}}{d \log x}\right)_{Breakdown}}{\left(\dfrac{d \log \overline{u}}{d \log x}\right)_{Plot}}$
Source flow; C_2 = 0.1; curve 1				
0.2	0.04	25.0	17.50	0.7
.25	.13	7.69	11.07	1.44
.3	.27	3.70	7.57	2.04
.4	.90	1.11	4.09	3.68
.5	2.36	.424	2.47	5.84
.6	7.17	.140	1.59	11.38
.65	15.5	.0645	1.28	19.9
Source flow; C_2 = 0.1; curve 2				
0.1	-0.92	-1.09	71.07	-65.39
.2	-.33	-3.03	17.50	-5.77
.3	-.25	-4.00	7.57	-1.89
.35	-.30	-3.33	5.46	-1.64
.4	-.38	-2.63	4.09	-1.55
.5	-.70	-1.43	2.47	-1.73
.6	-1.90	-.526	1.59	-3.02
Source flow; C_2 = 0.1; curve 3				
0.65	14.1	0.071	1.28	18.1
.7	8.70	.115	1.04	9.06
.75	2.38	.420	.840	2.00
.8	1.08	.926	.670	.723
.9	.45	2.22	.384	.173
Sink flow; C_2 = -0.1; curve 4				
0.2	-0.85	-1.18	-17.50	14.9
.3	-.70	-1.43	-7.57	5.3
.4	-.60	-1.67	-4.09	2.46
.5	-.49	-2.04	-2.47	1.21
.6	-.40	-2.50	-1.59	.635
.7	-.33	-3.03	-1.04	.344
.8	-.26	-3.85	-.670	.174

NACA

TABLE I.- Concluded

RATIOS OF $\left(\frac{p}{2\mu\frac{du}{dx}}\right)_{Plot}$ GIVEN BY PLOTTED SOLUTIONS OF NAVIER-STOKES EQUATION TO BREAKDOWN VALUES OF $\left(\frac{p}{2\mu\frac{du}{dx}}\right)_{Breakdown}$ OF NAVIER-STOKES EQUATION - Concluded

[Plotted solutions are given in figures 1 to 3.]

$\bar{u}$	$\phi' \equiv \frac{d \log x}{d \log \bar{u}}$	$\frac{1}{\phi'} \equiv \left(\frac{d \log \bar{u}}{d \log x}\right)_{Plot}$	$\left(\frac{d \log \bar{u}}{d \log x}\right)_{Breakdown}$	$\frac{\left(\frac{p}{2\mu\frac{du}{dx}}\right)_{Plot}}{\left(\frac{p}{2\mu\frac{du}{dx}}\right)_{Breakdown}} \equiv \frac{\left(\frac{d \log \bar{u}}{d \log x}\right)_{Breakdown}}{\left(\frac{d \log \bar{u}}{d \log x}\right)_{Plot}}$
Sink flow; C_2 = -0.1; curve 5				
0.15	1.40	0.714	-31.39	-43.9
.2	.20	5.00	-17.50	-3.50
.3	.18	5.56	-7.57	-1.36
.4	.25	4.00	-4.09	-1.02
.5	.35	2.86	-2.47	-.866
.6	.50	2.00	-1.59	-.794
.7	.76	1.32	-1.04	-.791
.8	1.18	.847	-.670	-.790
.9	2.33	.429	-.384	-.896
.95	5.50	.182	-.247	-1.36
Sink flow; C_2 = -0.1; curve 6				
0.1	-0.02	-50.0	-71.1	1.42
.11	-.06	-16.7	-58.7	3.52
.12	-.11	-9.09	-49.2	5.42
.13	-.17	-5.88	-41.9	7.12
.15	-.68	-1.47	-31.4	21.3
.17	-.89	-1.12	-24.4	21.7
Sink flow, C_2 = -0.1; curve 7				
0.46	3.3, -3.13	0.303,-0.319	-3.00	-9.89,9.38
.5	1.26,-1.05	.794, -.952	-2.47	-3.12,2.60
.55	1.08, -.68	.926,-1.47	-1.97	-2.13,1.34
.6	1.07, -.53	.935,-1.89	-1.59	-1.70, .841
.65	1.13, -.44	.885,-2.27	-1.28	-1.45, .565
.7	1.23, -.38	.813,-2.63	-1.04	-1.28, .396
.75	1.39, -.33	.719,-3.03	-.840	-1.17, .277
.8	1.65, -.26	.606,-3.85	-.670	-1.10, .174
.85	1.98, -.23	.505,-4.35	-.520	-1.03, .120
.9	2.60, -.19	.385,-5.26	-.384	-1.00, .0730
.95	5.60, -.12	.179,-8.33	-.247	-1.38, .0296

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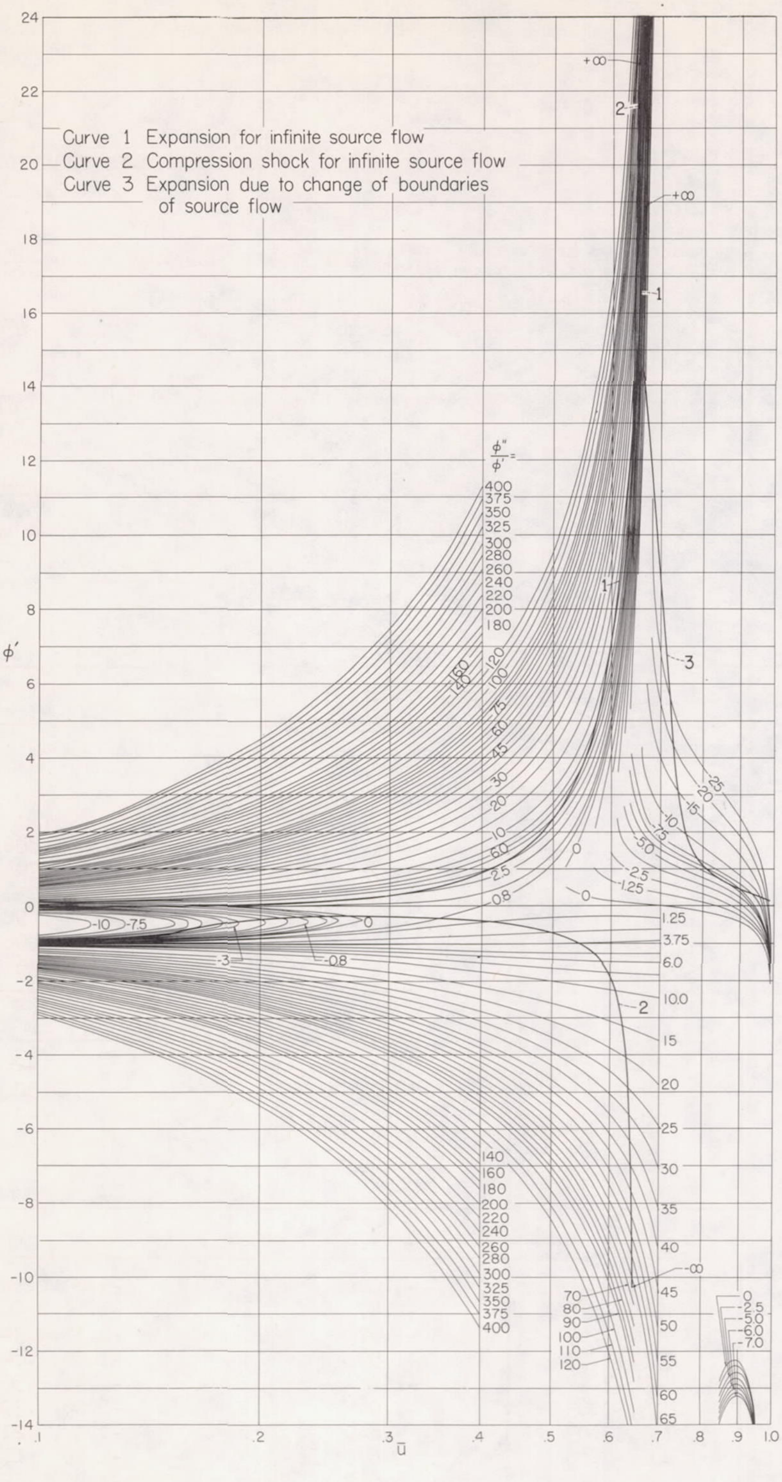


Figure 1.- Direction field and integration curves in plane of ϕ' plotted against $\bar{u}$ ($\bar{u}$ in log scale) for source flow with $C_2 = 0.1$. (A large-size print of this figure is enclosed in the inside pocket of the back cover page.)

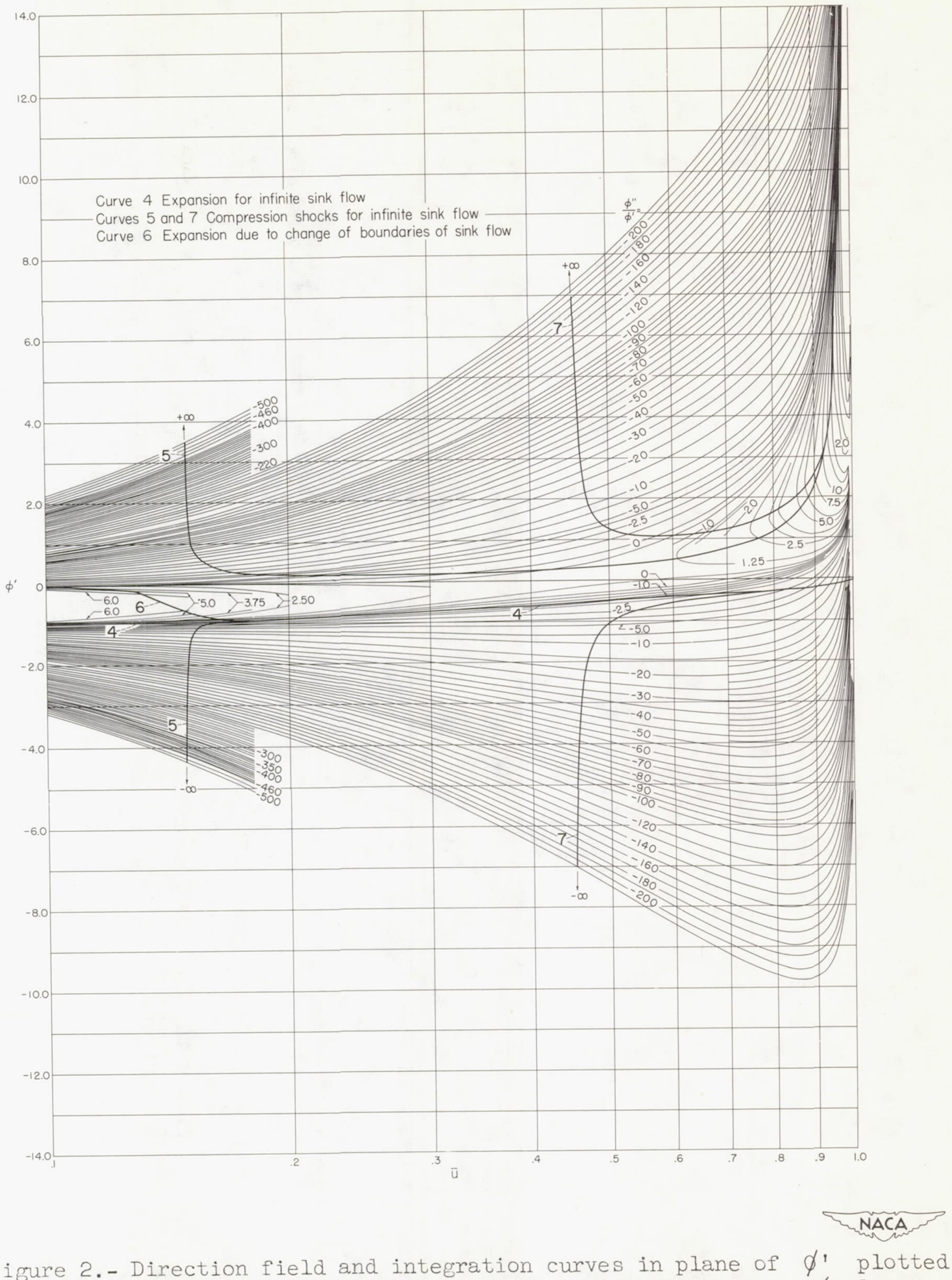


Figure 2.- Direction field and integration curves in plane of ϕ' plotted against $\bar{u}$ ($\bar{u}$ in log scale) for sink flow with $C_2 = -0.1$. (A large-size print of this figure is enclosed in the inside pocket of the back cover page.)

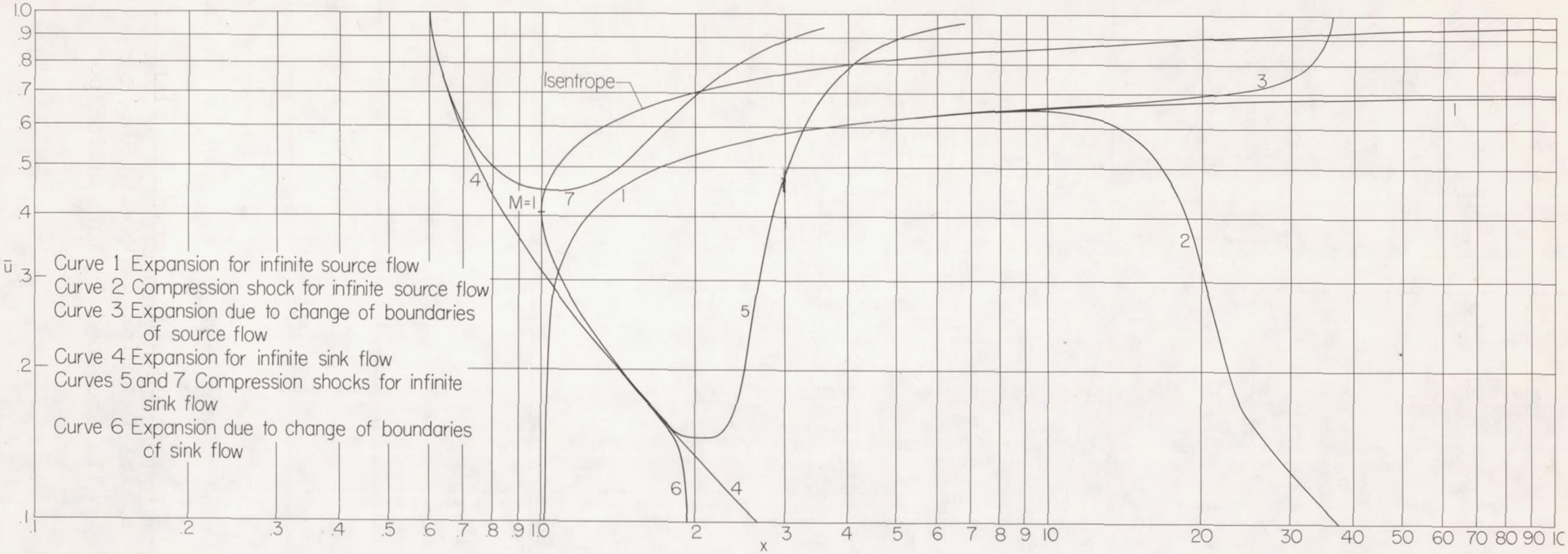
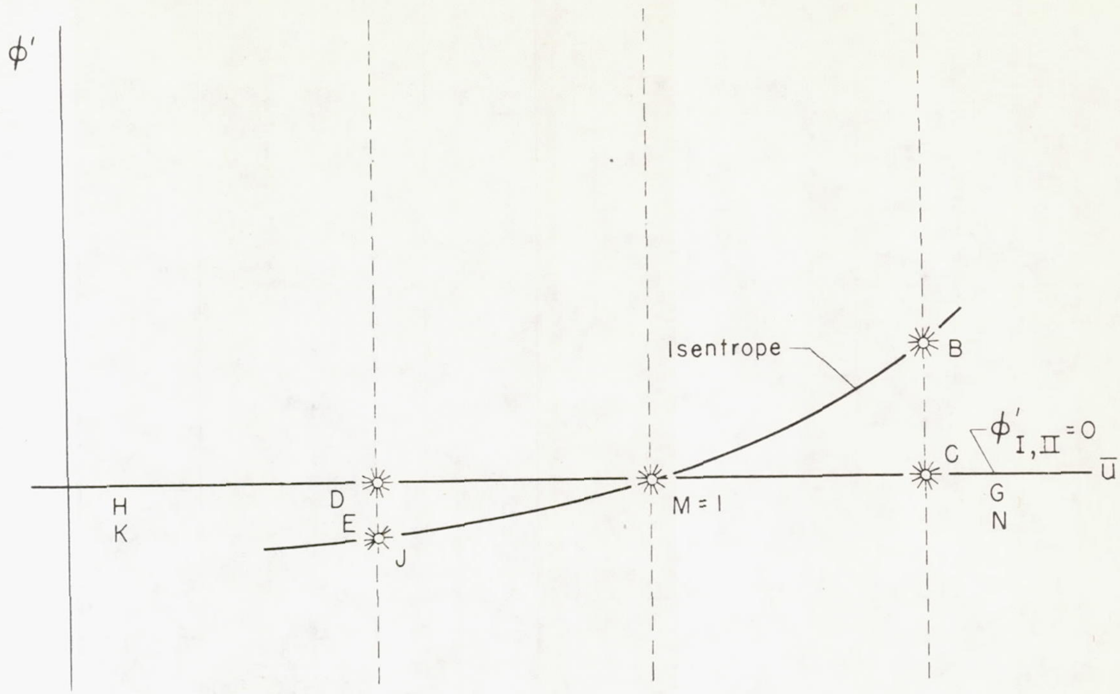
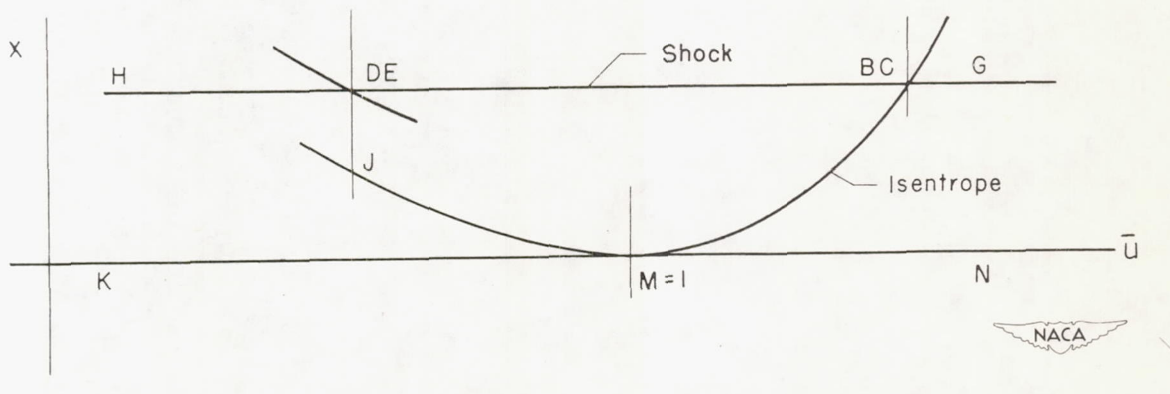


Figure 3.- Flow solutions in plane of $\bar{u}$ plotted against x (in log scale) for source flow with $C_2 = 0.1$ (curves 1, 2, and 3) and sink flow with $C_2 = -0.1$ (curves 4, 5, 6, and 7). (A large-size print of this figure is enclosed in the inside pocket of the back cover page.)



(a) Direction field and integration curves in plane of ϕ' plotted against $\bar{u}$ ($\bar{u}$ in log scale).



(b) Flow solutions in plane of $\bar{u}$ plotted against x (in log scale).

Figure 4.- Flow presentation when viscous effects of source or sink flow are zero, $C_2 = 0$.

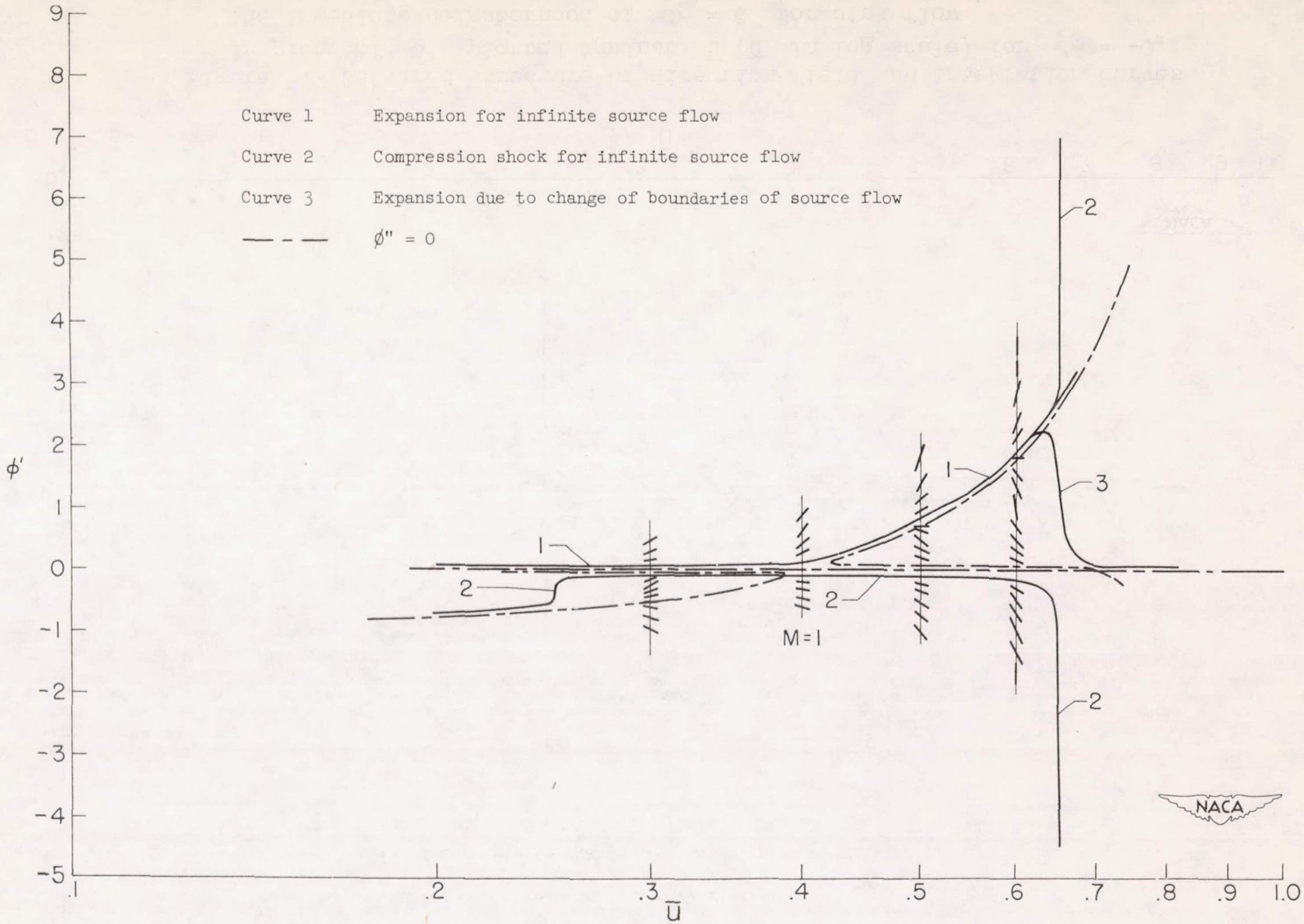


Figure 5.- Detailed structure of direction field and integration curves in plane of ϕ' plotted against $\bar{u}$ ($\bar{u}$ in log scale) for $C_2 = +0$, the immediate neighborhood of $C_2 = 0$ for source flow.

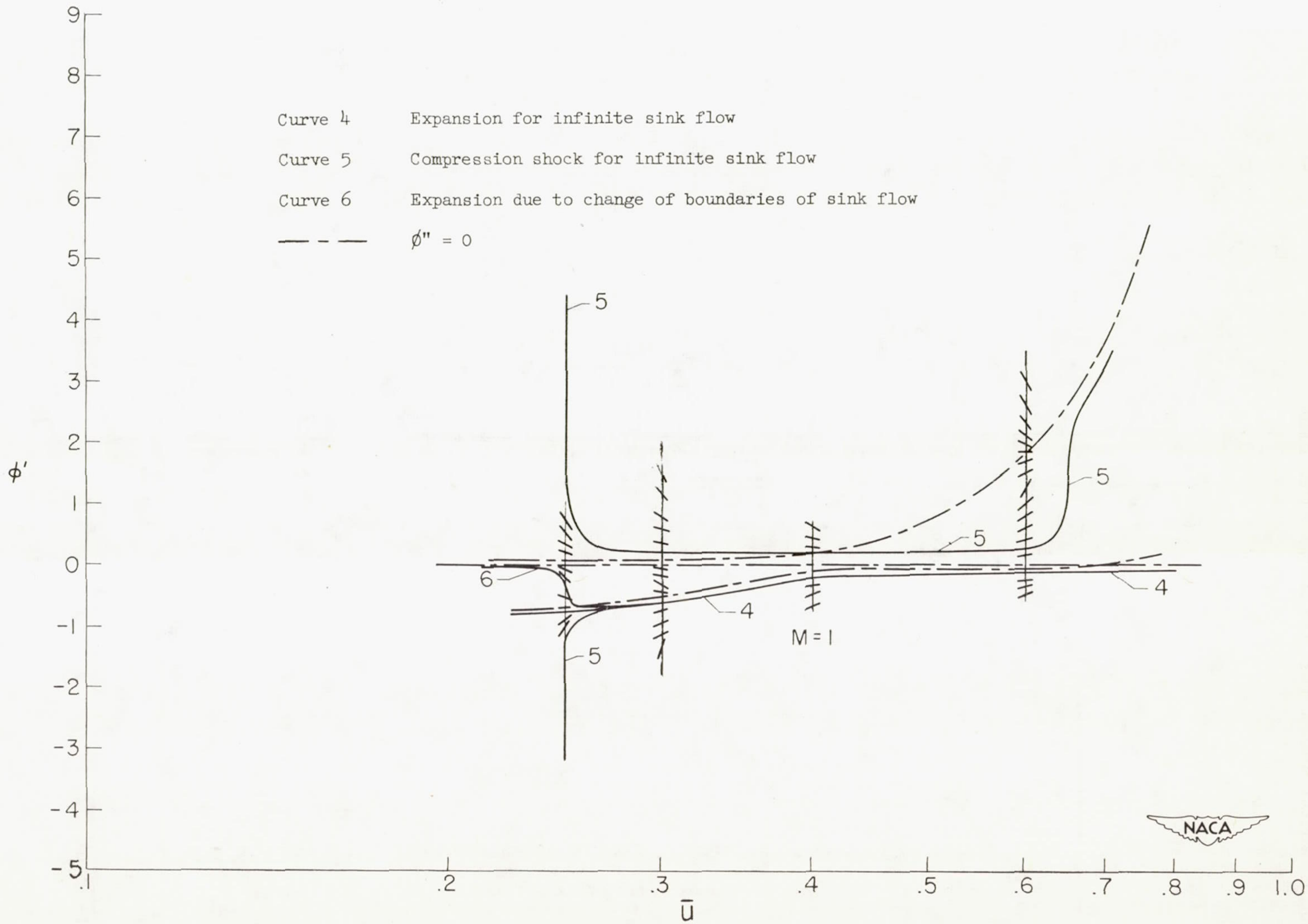
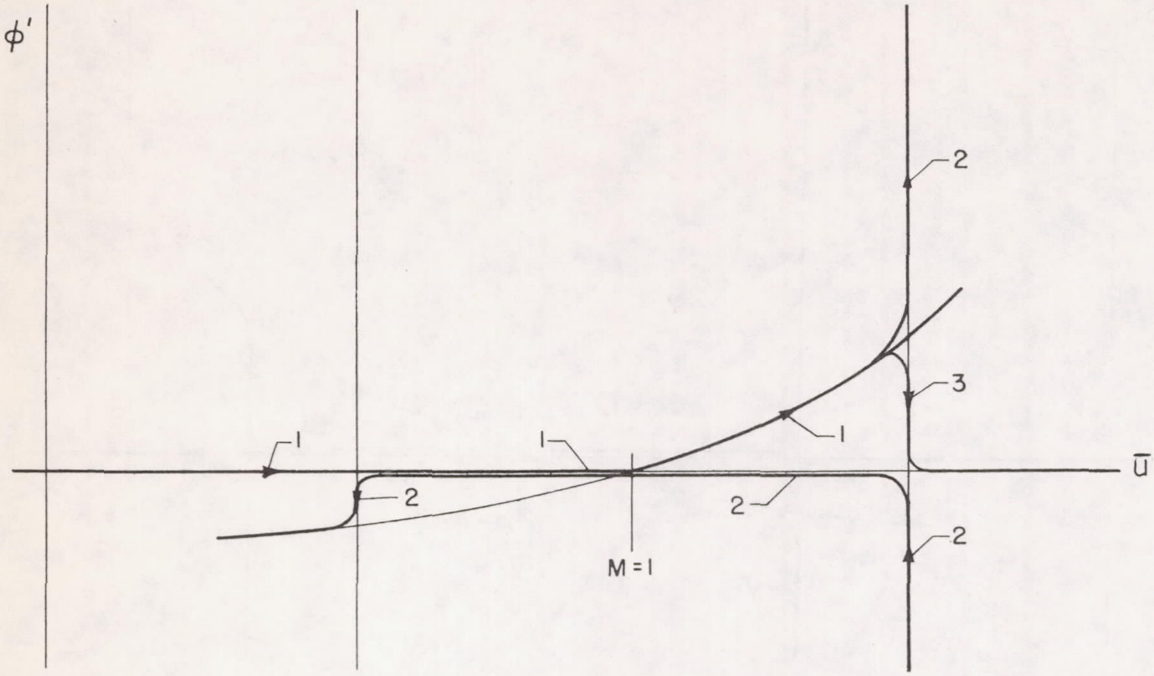


Figure 6.- Detailed structure of direction field and integration curves in plane of ϕ' plotted against $\bar{u}$ ($\bar{u}$ in log scale) for $C_2 = -0$, the immediate neighborhood of $C_2 = 0$ for sink flow.

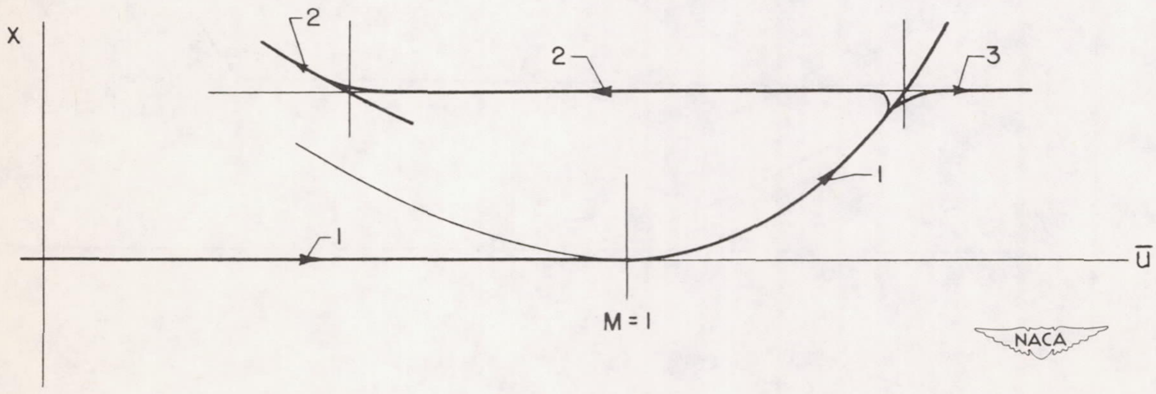
Curve 1 Expansion for infinite source flow

Curve 2 Compression shock for infinite source flow

Curve 3 Expansion due to change of boundaries of source flow



(a) Summary account of direction field and integration curves in plane of ϕ' plotted against $\bar{u}$ ($\bar{u}$ in log scale).



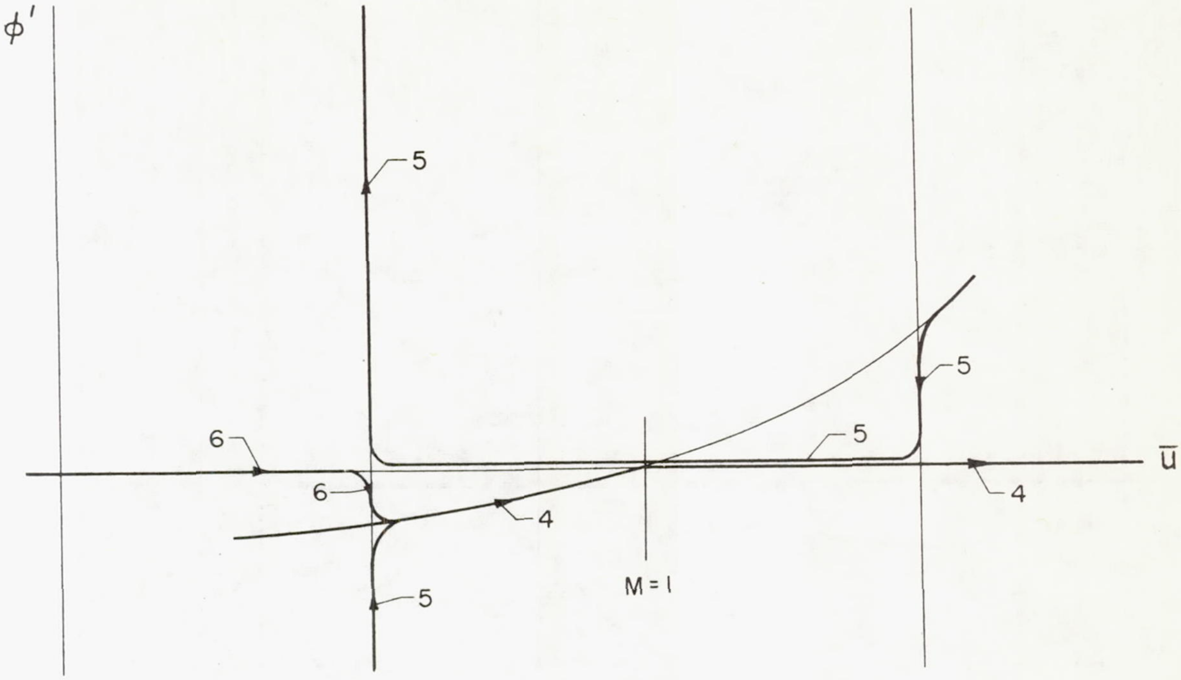
(b) Flow solutions in plane of $\bar{u}$ plotted against x (in log scale).

Figure 7.- Flow presentation of $C_2 = +0$, the immediate neighborhood of $C_2 = 0$ for source flow.

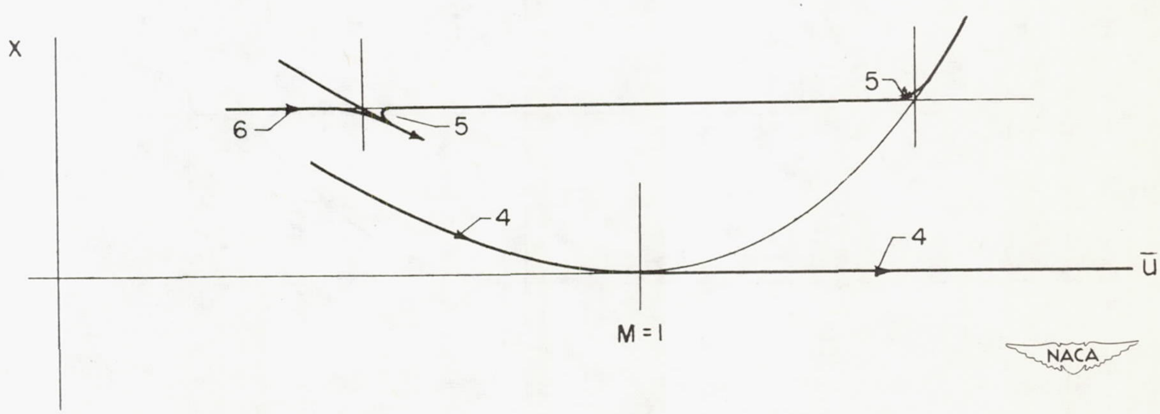
Curve 4 Expansion for infinite sink flow

Curve 5 Compression shock for infinite sink flow

Curve 6 Expansion due to change of boundaries of sink flow



(a) Summary account of direction field and integration curves in plane of ϕ' plotted against $\bar{u}$ ($\bar{u}$ in log scale).



(b) Flow solutions in plane of $\bar{u}$ plotted against x (in log scale).

Figure 8.- Flow presentation for $C_2 = -0$, the immediate neighborhood of $C_2 = 0$ for sink flow.